Vectors: components, magnitude and the scalar product: practice worksheet

IB Mathematics: analysis and approaches HL · AHL 3.12, 3.13 · 49 minutes · 41 marks

Question 1 (*no calculator*)

No calculator. The points A and B have coordinates $(2, -1, 3)$ and $(-1, 3, 3)$ respectively.

(a) Find $\overrightarrow{AB}$, giving your answer as a column vector. [1]

(b) Find $|\overrightarrow{AB}|$. [1]

(c) Hence write down a unit vector in the direction of $\overrightarrow{AB}$. [1]

Question 2 (*no calculator*)

No calculator. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$.

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. [2]

(b) The vector $\mathbf{w} = s\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, where $s \in \mathbb{R}$, is parallel to $\mathbf{u}$. Find s. [2]

Question 3 (*calculator*)

Calculator. The points A, B and C have coordinates $(1, -2, 4)$, $(4, 0, 3)$ and $(-1, 3, 6)$ respectively.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$. [2]

(c) Hence find $\angle BAC$, giving your answer in degrees to three significant figures. [2]

Question 4 (*calculator*)

Calculator. Two non-zero vectors are $\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(a) Find the exact values of $|\mathbf{p}|$ and $|\mathbf{q}|$. [2]

(b) Find the angle θ between $\mathbf{p}$ and $\mathbf{q}$, where $0° \leq \theta \leq 180°$. Give your answer in degrees to three significant figures. [3]

(c) Two intersecting straight lines have direction vectors $\mathbf{p}$ and $\mathbf{q}$. Hence find the acute angle between the lines, giving your answer in degrees to three significant figures. [2]

Question 5 (*no calculator*)

No calculator. The points A, B, C and D have coordinates $(1, -2, 2)$, $(7, 2, 6)$, $(6, -3, 4)$ and $(2, 3, 4)$ respectively. The point M is the midpoint of AB.

(a) Find the coordinates of M. [1]

(b) Show that C, M and D are collinear. [2]

(c) Show that AB is perpendicular to CD. [2]

(d) By finding appropriate side vectors, prove that the quadrilateral $ACBD$ is a rhombus. [3]

Question 6 (*no calculator*)

No calculator. This question links scalar products to quadratic equations and their discriminants. In two dimensions, $A = (-3, 0)$, $C = (3, 0)$ and $B = (t, kt + 5)$, where $t, k \in \mathbb{R}$.

(a) Show that $\overrightarrow{BA} \cdot \overrightarrow{BC} = (1 + k^2)t^2 + 10kt + 16$. [3]

(b) Hence determine the values of k for which the equation $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$ has exactly one real solution for t. [3]

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(c) For $k = \frac{4}{3}$, find the coordinates of the point B satisfying $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$. Hence find $\angle ABC$. [3]

(d) Determine all values of k for which there are two distinct positions of B satisfying $\angle ABC = 90°$. Each position must be distinct from both A and C. [4]

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Markscheme

Question 1

(a)

$$\overrightarrow{AB} = \begin{pmatrix} -1-2 \\ 3-(-1) \\ 3-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}. \quad \textbf{A1}$$

(b)

$$|\overrightarrow{AB}| = \sqrt{(-3)^2 + 4^2 + 0^2} = 5. \quad \textbf{A1}$$

(c)

$$\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}. \quad \textbf{A1}$$

Accept equivalent forms. FT a non-zero vector from (a), correctly normalised.

Question 2

(a)

$\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(2) + 2(0)$. **M1**

$\mathbf{u} \cdot \mathbf{v} = 0$. **A1**

Hence $\mathbf{u} \perp \mathbf{v}$. **AG**

(b)

Set $\mathbf{w} = \lambda\mathbf{u}$ and obtain $\lambda = \dfrac{3}{-1} = -3$. **M1**

$s = 2\lambda = -6$. **A1**

Accept using the third components to obtain the scalar multiplier.

Question 3

(a)

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}. \quad \textbf{A1}$$

$$\overrightarrow{AC} = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}. \quad \textbf{A1}$$

Accept equivalent base-vector notation.

(b)

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 3(-2) + 2(5) + (-1)(2)$. **M1**

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2$. **A1**

FT their vectors from (a).

(c)

$$\cos \angle BAC = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{2}{\sqrt{14}\sqrt{33}}. \quad \textbf{M1}$$

$\angle BAC = 84.7°$. **A1**

FT their vectors and scalar product where these give a defined angle. An answer only in radians does not earn the final A1.

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Question 4

(a)

$|\mathbf{p}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$. **A1**

$|\mathbf{q}| = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{21}$. **A1**

(b)

$\mathbf{p} \cdot \mathbf{q} = 2(-1) + (-3)(4) + 1(2)$. **M1**

$\cos\theta = \dfrac{-12}{\sqrt{14}\sqrt{21}} = -\dfrac{12}{\sqrt{294}}$. **A1**

$\theta = 134°$ to three significant figures. **A1**

Accept equivalent exact expressions for the cosine. An answer only in radians does not earn the final A1.

(c)

Use $180° - \theta$, retaining the unrounded value of θ. **M1**

The acute angle is $45.6°$. **A1**

Alternatively, use the absolute value of the scalar product in the cosine formula. FT their angle from (b) if the acute angle is selected correctly.

Question 5

(a)

$M = \left(\dfrac{1+7}{2}, \dfrac{-2+2}{2}, \dfrac{2+6}{2}\right) = (4, 0, 4)$. **A1**

(b)

Find $\overrightarrow{CM} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} -4 \\ 6 \\ 0 \end{pmatrix}$. **M1**

$\overrightarrow{CD} = 2\overrightarrow{CM}$. **A1**

Hence C, M and D are collinear. **AG**

Accept $\overrightarrow{CM} = \overrightarrow{MD}$ or another valid scalar-multiple argument using vectors with a common point.

(c)

$\overrightarrow{AB} \cdot \overrightarrow{CD} = 6(-4) + 4(6) + 4(0)$. **M1**

$\overrightarrow{AB} \cdot \overrightarrow{CD} = -24 + 24 = 0$. **A1**

Hence $AB \perp CD$. **AG**

(d)

Find $\overrightarrow{AC} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$, $\overrightarrow{BD} = -\overrightarrow{AC}$ and $\overrightarrow{DA} = -\overrightarrow{CB}$. **M1**

All four side vectors have magnitude $\sqrt{30}$. **A1**

The quadrilateral has four equal sides and is therefore a rhombus. **R1**

Alternatively, establish opposite sides as equal and parallel using the side vectors, then establish equality of adjacent side lengths.

Question 6

(a)

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Form $\overrightarrow{BA} = \begin{pmatrix} -3-t \\ -kt-5 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 3-t \\ -kt-5 \end{pmatrix}$. **M1**

$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-3-t)(3-t) + (-kt-5)^2 = t^2 - 9 + (kt+5)^2$. **A1**

Expand and collect terms to obtain $t^2 - 9 + k^2t^2 + 10kt + 25$. **A1**

$\overrightarrow{BA} \cdot \overrightarrow{BC} = (1+k^2)t^2 + 10kt + 16$. **AG**

Accept equivalent expansions.

(b)

Set the discriminant equal to zero: $(10k)^2 - 4(1+k^2)(16) = 0$. **M1**

$36k^2 - 64 = 0$, giving $k^2 = \frac{16}{9}$. **A1**

$k = \pm\frac{4}{3}$. **A1**

The coefficient $1 + k^2$ is positive for every real k, so the equation is always quadratic in t.

(c)

Substitute $k = \frac{4}{3}$ to obtain $(5t+12)^2 = 0$, so $t = -\frac{12}{5}$. **M1**

$B = \left(-\frac{12}{5}, \frac{4}{3}\left(-\frac{12}{5}\right) + 5\right) = \left(-\frac{12}{5}, \frac{9}{5}\right)$. **A1**

Both vectors are non-zero and their scalar product is zero, so $\angle ABC = 90°$. **A1**

Accept solving the repeated-root quadratic by another valid method.

(d)

Require two distinct real roots: $36k^2 - 64 > 0$. **M1**

Obtain $k < -\frac{4}{3}$ or $k > \frac{4}{3}$. **A1**

$B = A$ occurs when $t = -3$ and $k = \frac{5}{3}$; $B = C$ occurs when $t = 3$ and $k = -\frac{5}{3}$. These values must be excluded because one of the vectors is zero. **R1**

$|k| > \frac{4}{3}$, with $k \neq \pm\frac{5}{3}$. **A1**

Distinct roots give distinct points because the first coordinate of B is t. Accept an equivalent union of intervals.

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