

Vector equations of lines (intersecting, parallel and skew lines): practice worksheet

IB Mathematics: analysis and approaches HL · AHL 3.14-3.15 · 53 minutes · 44 marks

Question 1 (no calculator)

The points A and B have coordinates $(2, -1, 4)$ and $(5, 1, 0)$ respectively. The line L passes through A and B .

- (a) Find a vector equation of L . [2]
 (b) Write down the equation of L in Cartesian form. [1]

Question 2 (no calculator)

The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \quad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}.$$

- (a) Show that the point $P(3, 7, 2)$ lies on L_1 . [2]
 (b) Find the cosine of the acute angle between L_1 and L_2 . [2]

Question 3 (no calculator)

The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \quad L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mu \in \mathbb{R}.$$

- (a) Show that L_1 and L_2 are not parallel. [1]
 (b) Find the coordinates of the point of intersection of L_1 and L_2 . [4]
 (c) Find the acute angle between L_1 and L_2 , giving your answer in the form $\arccos k$ where $k \in \mathbb{Q}$. [2]

Question 4 (no calculator)

The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, s \in \mathbb{R} \quad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, t \in \mathbb{R}.$$

- (a) Show that L_1 and L_2 are not parallel. [1]
 (b) Show that L_1 and L_2 do not intersect, and hence deduce that L_1 and L_2 are skew. [4]
 (c) The line L_3 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}, t \in \mathbb{R}$, where $k \in \mathbb{R}$. Find the value of k for which L_1 and L_3 intersect. [3]

Question 5 (calculator)

Two drones, A and B , fly in straight lines with constant velocities. At time t seconds after launch, their position vectors, in metres relative to a fixed origin, are given by

$$\mathbf{r}_A = \begin{pmatrix} 0 \\ 5 \\ 10 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{r}_B = \begin{pmatrix} 28 \\ -9 \\ 7 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad t \geq 0.$$

- (a) Write down the speed of drone A . [1]
- (b) Find the acute angle between the flight paths of the two drones. [3]
- (c) Show that the flight paths of the two drones intersect, and find the coordinates of the point of intersection. [4]
- (d) Explain whether or not the two drones collide. [1]

Question 6 (*no calculator*)

The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \quad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R},$$

where $a \in \mathbb{R}$.

- (a) Show that L_1 and L_2 are not parallel for any value of a . [2]
- (b) Find the value of a for which L_1 and L_2 intersect, and find the coordinates of the point of intersection. [5]
- (c) Find the value of a for which L_1 and L_2 are perpendicular. [2]
- (d) In the case $a = -2$, find the Cartesian equation of the plane Π that contains both L_1 and L_2 . Give your answer in the form $px + qy + rz = d$ where $p, q, r, d \in \mathbb{Z}$. [4]

Markscheme

Question 1

(a)

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \text{ (or any scalar multiple)} \quad \text{M1}$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \quad \text{A1}$$

Award A0 if " $\mathbf{r} =$ " is not seen. Accept the position vector of B as the point and any non-zero scalar multiple of the direction vector.

(b)

$$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-4}{-4} \quad \text{A1}$$

Accept $\frac{4-z}{4}$ for the third fraction. Follow through from their direction vector in part (a).

Question 2

(a)

equating one component, e.g. $1 + \lambda = 3 \Rightarrow \lambda = 2$ M1

verifying the other two components: $3 + 2(2) = 7$ and $-2 + 2(2) = 2$, so P lies on L_1 A1
AG

All three components must be checked for the A1.

(b)

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2 - 2 + 4 = 4 \text{ and } |\mathbf{b}_1| = |\mathbf{b}_2| = 3 \quad \text{M1}$$

$$\cos \theta = \frac{4}{3 \times 3} = \frac{4}{9} \quad \text{A1}$$

Question 3

(a)

direction vectors are not scalar multiples of each other (e.g. $k = \frac{1}{2}$ from the x -components but $k = -1$ from the y -components), so not parallel R1

AG

A statement that the vectors are not parallel without justification scores R0.

(b)

equating components: $1 + \lambda = 1 + 2\mu$, $2 - \lambda = -1 + \mu$, $3 + 2\lambda = 8 - \mu$ M1

solving two equations: $\mu = 1, \lambda = 2$ A1

checking the third equation: $3 + 4 = 7 = 8 - 1$, so the lines intersect A1

point of intersection $(3, 0, 7)$ A1

Award M0 if the same parameter is used for both lines. Accept the answer as a position vector.

(c)

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -1, \quad |\mathbf{b}_1| = |\mathbf{b}_2| = \sqrt{6} \quad \text{M1}$$

$$\cos \theta = \frac{|-1|}{6} \Rightarrow \theta = \arccos\left(\frac{1}{6}\right) \quad \text{A1}$$

Award A0 for $\arccos\left(-\frac{1}{6}\right)$ as the final answer since the acute angle is required.

Question 4

(a)

direction vectors are not scalar multiples (e.g. $k = 2$ from the x -components, $k = -1$ from the y -components), hence not parallel R1

AG

(b)

equating components: $1 + 2s = 3 + t$, $s = 4 - t$, $2 + s = -1 + 2t$ M1

solving the first two equations: $t = 2$, $s = 2$ A1

substituting into the third equation: $2 + 2 = 4$ but $-1 + 4 = 3$, $4 \neq 3$, so there is no common point and the lines do not intersect R1

the lines are not parallel (from part (a)) and do not intersect, so they are skew R1
AG

Award the final R1 only if both conditions (not parallel and not intersecting) are explicitly stated. Any pair of equations may be solved first, with the remaining equation used for the contradiction.

(c)

the x - and y -components are unchanged, so $s = t = 2$ (or re-solving $1 + 2s = 3 + t$, $s = 4 - t$) M1

substituting into the z -components: $2 + 2 = -1 + 2k$ M1

$$k = \frac{5}{2} \quad \text{A1}$$

Note that $\begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}$ is never parallel to $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, so this value of k gives a unique point of intersection.

Question 5

(a)

$$\sqrt{4^2 + 2^2 + 1^2} = \sqrt{21} \quad (= 4.58 \text{ m s}^{-1}) \quad \text{A1}$$

Units are not required for the mark.

(b)

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = -4 + 6 + 1 = 3 \quad \text{M1}$$

$$\cos \theta = \frac{3}{\sqrt{21}\sqrt{11}} \quad \left(= \frac{3}{\sqrt{231}} \right) \quad \text{A1}$$

$$\theta = 78.6^\circ \quad (\text{accept } 1.37 \text{ radians}) \quad \text{A1}$$

Accept 78.62° . Award A1A0 for 101° unless the acute angle is subsequently stated.

(c)

using different parameters for the two paths: $4\lambda = 28 - \mu$, $5 + 2\lambda = -9 + 3\mu$, **M1**

$$10 + \lambda = 7 + \mu$$

solving two of the equations (e.g. $\mu = \lambda + 3$ then $5\lambda = 25$): $\lambda = 5$, $\mu = 8$ **A1**

checking the remaining equation: $5 + 10 = 15$ and $-9 + 24 = 15$, consistent, so the **R1**

paths intersect

point of intersection (20, 15, 15) **A1**

Award M0 if the same parameter t is used for both drones, as this only tests for a collision rather than an intersection of paths.

(d)

A is at the point of intersection when $t = 5$ and B when $t = 8$; since the times are different the drones do not collide **R1**

Follow through from their parameter values in part (c).

Question 6

(a)

supposing $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = k \begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix}$ for some k **M1**

y -components give $k = -1$, z -components give $k = \frac{1}{2}$, contradiction independent of a , so not parallel for any a **R1**

AG

The argument must not depend on the x -component, since that is the only component involving a .

(b)

equating components: $1 + 2\lambda = 3 + a\mu$, $2 - 2\lambda = 2\mu$, $-1 + \lambda = 3 + 2\mu$ **M1**

from the y -components $\lambda = 1 - \mu$ **A1**

substituting into the z -components: $-\mu = 3 + 2\mu \Rightarrow \mu = -1$, $\lambda = 2$ **A1**

substituting into the x -components: $5 = 3 - a \Rightarrow a = -2$ **A1**

point of intersection (5, -2, 1) **A1**

Since the lines are never parallel (part (a)), $a = -2$ gives a unique point of intersection. Accept the point as a position vector.

(c)

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix} = 0 \Rightarrow 2a - 4 + 2 = 0$$

$$a = 1$$

(d)

$$\text{normal vector } \mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{n} = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} \text{ (or } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix})$$

substituting a point on the plane, e.g. (1, 2, -1) or (3, 0, 3) or (5, -2, 1): $x + y = 1 + 2$ **M1**

$x + y = 3$ (accept $6x + 6y = 18$) **A1**

Accept any correct integer scalar multiple. Award the first M1 for any valid attempt at the cross product, including determinant notation.