Sequences and series: arithmetic, geometric and financial applications: practice worksheet

IB Mathematics: analysis and approaches HL · SL 1.2, 1.3, 1.4, 1.8 · 49 minutes · 41 marks

Question 1 (*no calculator*)

No calculator. An arithmetic sequence has third term 14 and eighth term 39.

(a) Find the common difference. [2]

(b) Hence find the twelfth term. [1]

Question 2 (*no calculator*)

No calculator. A geometric sequence has first term 18 and common ratio $\frac{2}{3}$.

(a) Find the sixth term. [2]

(b) Write down the sum of the first four terms using sigma notation. [1]

(c) Find the exact value of this sum. [1]

Question 3 (*no calculator*)

No calculator. A deposit earns simple interest, credited at the end of each year. No money is added or withdrawn. Let B_n be the balance, in pounds, immediately after interest is credited for year n. The balances satisfy $B_3 = 1340$ and $B_8 = 1740$.

(a) Find the interest credited each year and the original deposit. [3]

(b) Let T_n be the sum of the balances recorded at the ends of the first n years. Write down an expression for T_n using sigma notation. [1]

(c) Determine the least positive integer n for which $T_n > 20000$. [3]

Question 4 (*no calculator*)

No calculator. A real geometric sequence has second term 12 and fifth term $-\frac{3}{2}$. Let S_n denote the sum of its first n terms.

(a) Find the common ratio and the first term. [3]

(b) Explain why the infinite series converges and find its sum S_∞. [2]

(c) Determine the least positive integer n for which $|S_n - S_\infty| < \frac{1}{4}$. [2]

Question 5 (*calculator*)

Calculator. Leena deposits £2500 into an account paying a nominal annual interest rate of 6%, compounded monthly. There are no further deposits or withdrawals. Interest is credited at the end of each month.

(a) Find the balance immediately after the twenty-fourth monthly interest payment. Give your answer to the nearest penny. [3]

(b) Determine the least number of complete months required for the balance to reach at least £3000. [3]

(c) Separately, Leena buys equipment for £3200. Its value depreciates by 15% at the end of each year. Determine the least number of complete years after which its value is below £2000. [2]

Question 6 (*no calculator*)

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No calculator. Three expressions, in the order given, are $10 - 2p$, $8 - 4p$ and $p + 1$, where $p \in \mathbb{R}$. This question links sequence conditions with quadratic equations and the maximum of a quadratic expression.

(a) Suppose these expressions are the first three terms of a geometric sequence. Show that $p^2 - 4p + 3 = 0$. [3]

(b) Hence find the possible values of p and the corresponding common ratios. [3]

(c) Determine which value of p gives a convergent infinite geometric series. State a reason and find its sum. [2]

(d) Now suppose instead that the three expressions are the first three terms of an arithmetic sequence. Find p. Hence determine the maximum possible value of S_n, the sum of the first n terms, for positive integers n, and state where this maximum occurs. [4]

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Markscheme

Question 1

(a)

Use $39 - 14 = (8-3)d$. **M1**

$d = 5$. **A1**

Accept an equivalent method using simultaneous equations.

(b)

$u_{12} = 14 + 9(5) = 59$. **A1**

FT their common difference using $u_{12} = 14 + 9d$.

Question 2

(a)

Use $u_6 = 18\left(\frac{2}{3}\right)^5$. **M1**

$u_6 = \frac{64}{27}$. **A1**

Accept equivalent exact forms.

(b)

$$\sum_{k=1}^{4} 18\left(\frac{2}{3}\right)^{k-1}.$$ **A1**

Accept equivalent indexing with matching limits and exponent.

(c)

$S_4 = 18 + 12 + 8 + \frac{16}{3} = \frac{130}{3}$. **A1**

Accept use of the geometric series formula.

Question 3

(a)

Let P be the original deposit and I the annual interest; use $P+3I = 1340$ and $P+8I = 1740$. **M1**

$I = 80$. **A1**

$P = 1340 - 3(80) = 1100$. **A1**

Accept finding $I = (1740 - 1340)/5$ *directly.*

(b)

$$T_n = \sum_{k=1}^{n} (1100 + 80k).$$ **A1**

FT their values of P and I. Accept equivalent indexing.

(c)

Use $T_n = \frac{n}{2}[2(1180) + (n-1)80] = 40n^2 + 1140n$. **M1**

$T_{12} = 19440$ and $T_{13} = 21580$. **A1**

Since the balances are positive, T_n increases; therefore the least integer is $n = 13$. **A1**

Accept solving the quadratic inequality and checking the adjacent integers. FT consistent earlier values if the resulting sums increase.

Question 4

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(a)

Use $r^3 = \frac{-3/2}{12} = -\frac{1}{8}$.	**M1**
$r = -\frac{1}{2}$.	**A1**
$u_1 = \frac{12}{-1/2} = -24$.	**A1**

Accept equivalent simultaneous equations.

(b)

$\lvert r\rvert = \frac{1}{2} < 1$, so the infinite geometric series converges.	**R1**
$S_\infty = \frac{-24}{1-(-1/2)} = -16$.	**A1**

The reason must refer to the magnitude of the ratio. FT a convergent ratio and first term from part (a).

(c)

Use $\lvert S_n - S_\infty\rvert = 16\left(\frac{1}{2}\right)^n$, giving $2^n > 64$.	**M1**
At $n = 6$ the error equals $\frac{1}{4}$, while at $n = 7$ it equals $\frac{1}{8}$; hence $n = 7$.	**A1**

Accept comparison of successive error values. The strict inequality excludes $n = 6$.

Question 5

(a)

Use the monthly interest rate $\frac{0.06}{12} = 0.005$.	**M1**
The balance is $2500(1.005)^{24}$.	**A1**
£2817.90.	**A1**

Accept a finance solver with PV = -2500, PMT = 0, N = 24, I% = 6 and P/Y = C/Y = 12, with inputs stated.

(b)

Set up $2500(1.005)^n \geq 3000$, or use an equivalent finance solver or table.	**M1**
After 36 months the balance is approximately £2991.70; after 37 months it is approximately £3006.66.	**A1**
The least number of complete months is 37.	**A1**

For the middle mark, accept a continuous solution of approximately 36.6 months. Finance solver inputs: PV = -2500, FV = 3000, PMT = 0, I% = 6, P/Y = C/Y = 12. FT their monthly multiplier if greater than 1.

(c)

Use $3200(0.85)^t < 2000$, or an equivalent table of annual values.	**M1**
The values after 2 and 3 years are £2312 and £1965.20 respectively, so the least number is 3 years.	**A1**

Accept solving the equality and rounding upwards to the next complete year.

Question 6

(a)

Use $(8-4p)^2 = (10-2p)(p+1)$.	**M1**
Expand to obtain $64 - 64p + 16p^2 = -2p^2 + 8p + 10$.	**A1**
Rearrange to obtain $18p^2 - 72p + 54 = 0$.	**A1**
Divide by 18 to obtain $p^2 - 4p + 3 = 0$.	**AG**

Explicit substitution and algebra are required; checking particular values of p is insufficient. AG carries 0 marks.

(b)

$(p-1)(p-3) = 0$, so $p = 1$ or $p = 3$.	**A1**
For each value, evaluate $r = \frac{8-4p}{10-2p}$.	**M1**
$p = 1$ gives $r = \frac{1}{2}$, and $p = 3$ gives $r = -1$.	**A1**

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Accept ratios found using the third and second terms.

(c)

Select $p = 1$ because its ratio satisfies $|r| < 1$; reject $p = 3$ because $|-1| = 1$. **R1**

$S_\infty = \frac{8}{1-1/2} = 16$. **A1**

FT a convergent geometric sequence obtained consistently in part (b).

(d)

Use $2(8-4p) = (10-2p) + (p+1)$. **M1**

$p = \frac{5}{7}$, giving $u_1 = \frac{60}{7}$ and $d = -\frac{24}{7}$. **A1**

Use $S_n = \frac{n}{2}\left[\frac{120}{7} - (n-1)\frac{24}{7}\right] = \frac{108}{7} - \frac{12}{7}(n-3)^2$. **M1**

The maximum is $\frac{108}{7}$, attained only at $n = 3$. **A1**

Alternatively, after finding p, award the second M1 for identifying that $u_3 = 12/7 > 0$, $u_4 = -12/7 < 0$ *and the terms continue decreasing. FT consistent arithmetic parameters if a finite maximum exists.*

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