Sequences and series: arithmetic, geometric and financial applications: study notes

IB Mathematics: analysis and approaches HL · SL 1.2, 1.3, 1.4, 1.8

A sequence is an ordered list of numbers generated by a rule; a series is the sum of its terms. This topic covers the two standard rules, arithmetic (add a constant d) and geometric (multiply by a constant r), their term and sum formulae, sigma notation, the infinite geometric sum, and the money contexts built on them: simple interest, compound interest with several compounding periods per year, and annual depreciation. Exam questions are short and formulaic: find d or r from two pieces of information, find n for which a sum or value passes a target, write a sum in sigma notation, or justify convergence with $|r| < 1$.

1. Arithmetic sequences and series

An arithmetic sequence has a constant common difference $d = u_{n+1} - u_n$. The nth term is $u_n = u_1 + (n-1)d$ and the sum of the first n terms is $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$. Both are in the booklet; what you must learn is how to use them. Two given terms give two linear equations in u_1 and d; subtracting them gives d at once because $u_p - u_q = (p-q)d$. A term and a sum also give a pair of simultaneous equations. Note that $u_n = S_n - S_{n-1}$ for $n \geq 2$, and that for $d \neq 0$, S_n is a quadratic in n with no constant term (if $d = 0$ it is simply $S_n = nu_1$), so a condition like $S_n = 0$ or $S_n > 100$ becomes a quadratic equation or inequality in n. Simple interest is the standard application: a principal P at r% per year grows by the fixed amount $\frac{Pr}{100}$ each year, so the values after each year form an arithmetic sequence with $u_1 = P + \frac{Pr}{100}$ and $d = \frac{Pr}{100}$. Be careful with the term index: the value after n years is $P + \frac{nPr}{100}$, which is u_n only if u_1 is the value after the first year.

2. Sigma notation

Sigma notation is not in the booklet. $\sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n = S_n$. The variable under the sign is a dummy index; the lower and upper limits give the first and last values of the index. Count the number of terms as (upper limit − lower limit +1), so $\sum_{k=3}^{12}(2k+1)$ has 10 terms, not 12. To evaluate it, either write it as a difference $S_{12} - S_2$ of the full series, or treat it as its own arithmetic series with first term 7, last term 25 and 10 terms, giving $\frac{10}{2}(7 + 25) = 160$. To write a given series in sigma notation, identify u_1 and d (or r), write the general term in terms of k and fix the limits. For an infinite geometric series the upper limit is ∞: $\sum_{k=1}^{\infty} u_1 r^{k-1}$. Examiners accept any correct dummy variable, but the expression inside must produce exactly the listed terms.

3. Geometric sequences and series

A geometric sequence has a constant common ratio $r = \frac{u_{n+1}}{u_n}$, with $r \neq 0$. The nth term is $u_n = u_1 r^{n-1}$ and $S_n = \frac{u_1(r^n-1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$ for $r \neq 1$ (both in the booklet; the second form is convenient when $|r| < 1$). Two terms give r by division: $\frac{u_p}{u_q} = r^{p-q}$. If $p - q$ is even there are two possible values of r, positive and negative, and the question will usually tell you which to take or ask for both. When three consecutive terms are given as linear expressions in a parameter x, the arithmetic condition $u_2 - u_1 = u_3 - u_2$ gives a linear equation, while the geometric condition $u_2^2 = u_1 u_3$ gives a quadratic in x; reject any solution that makes a term zero. Applications: compound growth of money, a population growing by a fixed percentage each year ($r = 1 + \frac{p}{100}$), and depreciation ($r = 1 - \frac{p}{100}$). Finding n from a target value needs logarithms: from $r^n > c$ take ln of both sides, remembering that if $0 < r < 1$ then $\ln r < 0$ and the inequality reverses. On a calculator paper, a table of values of u_n or S_n on the GDC is an accepted method provided you write down the two values either side of the target.

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4. Infinite geometric series and convergence

If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$, so $S_n \to \frac{u_1}{1-r}$. The booklet gives $S_\infty = \frac{u_1}{1-r}$ with the condition $|r| < 1$, but you must be able to state the condition in words and use it as a reason: a geometric series with $u_1 \neq 0$ converges if and only if $-1 < r < 1$, endpoints excluded. The R mark in these questions is for writing this condition and applying it, for example when r is an expression such as $\frac{x-2}{3}$, leading to the strict interval $-1 < x < 5$. A recurring decimal is a geometric series: $0.\dot{2}\dot{7} = 0.27 + 0.0027 + \ldots$ with $u_1 = \frac{27}{100}$, $r = \frac{1}{100}$, so it equals $\frac{27/100}{99/100} = \frac{3}{11}$. When both S_∞ and one term are known, eliminating u_1 gives a quadratic in r; check each root against $|r| < 1$ before accepting it. Describe the behaviour of the partial sums if asked: for $0 < r < 1$ they approach the limit from one side (increasing if $u_1 > 0$, decreasing if $u_1 < 0$), for $-1 < r < 0$ they alternate above and below it.

5. Financial applications: compound interest and depreciation

The booklet formula is $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$, where PV is the present value, r the annual nominal rate as a percentage, k the number of compounding periods per year ($k = 1$ yearly, 2 half-yearly, 4 quarterly, 12 monthly) and n the number of years. Each period multiplies the balance by $1 + \frac{r}{100k}$, so the exponent counts periods, not years. Depreciation is not in the booklet: a value falling by $p\%$ per year is $V_n = V_0\left(1 - \frac{p}{100}\right)^n$, the same shape with a minus sign, so learn it. Typical tasks: find FV to two decimal places; find the rate or the initial amount by rearranging; find the number of years (or months) for the value to exceed, or fall below, a target. For the last type set up the inequality, solve with logarithms or by the finance solver, then round up to the next whole period and state the answer in the units the question demands (complete years, months). An inflation adjustment compares two growth factors: the real growth factor is $\frac{1+r/100}{1+i/100}$, which is close to, but not equal to, $1 + \frac{r-i}{100}$.

6. Technology: GDC, finance solver and spreadsheets

On Paper 2 the finance solver on the GDC is an accepted method. Its inputs are the number of periods N, the annual rate $I\%$, the present value PV, the payment PMT (zero for a lump sum), the future value FV, and the compounding settings. Sign convention matters: money paid in and money received have opposite signs, so a deposit of 5000 with $PV = -5000$ gives a positive FV. Write the inputs out in full on the page: a visible, correct setup is what the method mark rewards, and if the final answer is wrong the listed inputs are the only evidence that the method was right. A sequence or table on the GDC (or a spreadsheet with a cell formula such as previous cell times 1.003) is equally acceptable for 'how many years until' questions; show the two consecutive rows that bracket the target. Graphing software showing $y = S_x$ against x helps to see where an arithmetic sum is maximised: the vertex of the parabola need not fall at an integer, and when it does not you must test the integers on each side rather than rounding. Paper 1 is a non-calculator paper, so technology is for revision only there: use it at home to check your exact Paper 1-style work, but in the exam the exact fraction or surd form, reached by hand, is what earns the marks.

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Formulas

$u_n = u_1 + (n-1)d$	in the booklet
$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$	in the booklet
$u_n = u_1 r^{n-1}$	in the booklet
$S_n = \frac{u_1(r^n-1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$	in the booklet
$S_\infty = \frac{u_1}{1-r}$, $\lvert r\rvert < 1$	in the booklet
$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$	in the booklet
$V_n = V_0\left(1 - \frac{p}{100}\right)^n$ (depreciation)	learn it
$\sum_{k=1}^{n} u_k = S_n$, number of terms = upper − lower +1	learn it
Convergence: a geometric series with $u_1 \neq 0$ converges if and only if $-1 < r < 1$	learn it
$r^n > c \Rightarrow n \ln r > \ln c$ (for $r, c > 0$); divide by $\ln r$ and reverse the inequality if $0 < r < 1$	learn it
$u_n = S_n - S_{n-1}$ for $n \geq 2$	learn it

Worked examples

Example 1. (Paper 1 style, 8 marks) An arithmetic sequence has $u_4 = -3$ and $u_{10} = 9$. (a) Find the common difference and the first term. (b) Find the value of n, with $n > 1$, for which $S_n = 0$. (c) Find the least value of n for which $S_n > 100$.

1. (a) Subtract the term formulae: $u_{10} - u_4 = 6d$, so $6d = 12$ and $d = 2$ (M1 A1).
2. Then $u_4 = u_1 + 3d$ gives $-3 = u_1 + 6$, so $u_1 = -9$ (A1).
3. (b) $S_n = \frac{n}{2}(2(-9) + (n-1)(2)) = \frac{n}{2}(2n - 20) = n(n-10)$ (M1 A1).
4. $n(n-10) = 0$ with $n > 1$ gives $n = 10$ (A1). Check: the terms $-9, -7, \ldots, 9$ cancel in pairs.
5. (c) $n(n-10) > 100$, so $n^2 - 10n - 100 > 0$ (M1). The positive root of $n^2 - 10n - 100 = 0$ is $n = 5 + \sqrt{125} = 5 + 5\sqrt{5} \approx 16.2$.
6. Since n is a positive integer and the quadratic is positive beyond its larger root, the least n is 17. Check: $S_{16} = 16 \times 6 = 96$, $S_{17} = 17 \times 7 = 119$ (A1).

Answer: (a) $d = 2$, $u_1 = -9$. (b) $n = 10$. (c) $n = 17$.

Example 2. (Paper 1 style, 8 marks) The sum to infinity of a geometric series is 48 and its second term is 9. (a) Show that the common ratio r satisfies $16r^2 - 16r + 3 = 0$. (b) Find the two possible values of r and the corresponding first terms, stating why both are valid. (c) For the case with the larger first term, write the series using sigma notation.

1. (a) From the booklet, $S_\infty = \frac{u_1}{1-r} = 48$, so $u_1 = 48(1-r)$ (M1). The second term is $u_1 r = 9$ (A1).
2. Substitute: $48r(1-r) = 9$, so $48r - 48r^2 - 9 = 0$. Divide by -3: $16r^2 - 16r + 3 = 0$ (A1 AG). Write every line; the final mark of a 'show that' is usually withheld unless the expansion is shown.
3. (b) Factorise: $(4r-1)(4r-3) = 0$, so $r = \frac{1}{4}$ or $r = \frac{3}{4}$ (M1 A1).
4. Both satisfy $\lvert r\rvert < 1$, so both series converge and both are valid (R1).
5. First terms: $u_1 = 48\left(1 - \frac{1}{4}\right) = 36$ when $r = \frac{1}{4}$, and $u_1 = 48\left(1 - \frac{3}{4}\right) = 12$ when $r = \frac{3}{4}$ (A1).

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6. (c) The larger first term is 36 with $r = \frac{1}{4}$: the series is $\sum_{k=1}^{\infty} 36\left(\frac{1}{4}\right)^{k-1}$ (A1).

Answer: (a) Shown. (b) $r = \frac{1}{4}$, $u_1 = 36$ or $r = \frac{3}{4}$, $u_1 = 12$; both valid because $|r| < 1$. (c) $\sum_{k=1}^{\infty} 36\left(\frac{1}{4}\right)^{k-1}$.

Example 3. (Paper 2 style, 9 marks) Maria invests 5000 euros in an account paying a nominal annual interest rate of 3.6%, compounded monthly. (a) Find the value of the investment after 5 years, to the nearest cent. (b) Find the number of complete years required for the investment to exceed 8000 euros. (c) At the same time Maria buys a car for 24000 euros which depreciates by 15% per year. Find the value of the car after 4 years, to the nearest euro.

1. (a) Use $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$ with $PV = 5000$, $r = 3.6$, $k = 12$, $n = 5$: $FV = 5000\left(1 + \frac{3.6}{1200}\right)^{60} = 5000(1.003)^{60}$ (M1 A1).
2. $FV = 5984.47$ euros (A1). On the finance solver: $N = 60$, $I\% = 3.6$, $PV = -5000$, $PMT = 0$, $P/Y = C/Y = 12$; write these inputs down to earn the method mark.
3. (b) Require $5000(1.003)^{12n} > 8000$, so $(1.003)^{12n} > 1.6$ (M1 A1).
4. Take logarithms: $12n \ln 1.003 > \ln 1.6$, so $n > \frac{\ln 1.6}{12 \ln 1.003} = 13.07\ldots$ (A1). Since $\ln 1.003 > 0$ the inequality does not reverse.
5. The investment first exceeds 8000 euros during the 14th year, so 14 complete years are needed (A1). Check with a table: after 13 years the value is 7978.40, after 14 years it is 8270.40.
6. (c) Depreciation: $V = 24000(1 - 0.15)^4 = 24000(0.85)^4$ (M1). $0.85^4 = 0.52200625$, so $V = 12528.15 \approx 12528$ euros (A1).

Answer: (a) 5984.47 euros. (b) 14 years. (c) 12528 euros.

Common errors

- Confusing the term number with the exponent: the nth geometric term is $u_1 r^{n-1}$, not $u_1 r^n$, while the value after n years of compound growth is $PV(1+i)^n$. Decide first whether u_1 is the starting value or the value after one period.
- Using the number of years as the exponent when compounding is monthly: with $k = 12$ the exponent is $12n$ and the rate per period is $\frac{r}{1200}$. Write the booklet formula with all four quantities labelled before substituting.
- Claiming convergence without stating $|r| < 1$, or giving the interval with closed endpoints. The R mark needs the strict condition written down and checked against each candidate value of r.
- Rounding a continuous answer the wrong way when n must be an integer: a bound such as $n > 13.07$ means $n = 14$, and the integer n that maximises a sum is found by testing both integers either side of the vertex, not by rounding the vertex.
- Reversing, or failing to reverse, the inequality when dividing by $\ln r$: if $0 < r < 1$ then $\ln r$ is negative and the direction changes. Alternatively keep everything positive by writing $\left(\frac{1}{r}\right)^n$.
- Using rounded intermediate values such as $(1.003)^{60} \approx 1.20$ and then quoting a final answer to the nearest cent. Keep full calculator accuracy until the last line and round only once.

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Before the exam, can you...

- Find d or r from any two given terms by subtracting or dividing the term formulae, and solve the resulting pair of simultaneous equations?
- Convert a verbal condition ('first exceeds', 'is zero', 'falls below half') into an equation or inequality in n, solve it exactly or with logarithms, and justify the integer answer?
- Write a given series in sigma notation with correct limits and general term, and count the number of terms in a sum written in sigma notation?
- State the condition $|r| < 1$ in words, apply it to a ratio containing a parameter to obtain a strict interval, and use S_∞ for recurring decimals?
- Use the compound interest formula with k periods per year and the depreciation formula, and record finance solver inputs or a GDC table so the method is visible?

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