Rigid body mechanics: torque, rotation and angular momentum: study notes

IB Physics HL · A.4 (HL)

This topic extends Newtonian mechanics from point particles to extended bodies that can rotate. Every linear quantity has a rotational twin: force becomes torque, mass becomes moment of inertia, momentum becomes angular momentum, and the kinematic and energy equations carry over by analogy. The exam tests you with short Paper 1 items on torque, moments of inertia of point masses and conservation of angular momentum, and with a longer Paper 2 question on a body rolling down an incline, a flywheel or pulley driven by a torque, or a skater or disc whose moment of inertia changes.

1. Torque and couples

The torque (moment) of a force F about an axis is $\tau = Fr\sin\theta$, where r is the distance from the axis to the point where the force acts and θ is the angle between the force and the line joining the axis to that point. Equivalently, torque equals force multiplied by the perpendicular distance from the axis to the line of action of the force. The SI unit is N m (not the joule, even though the dimensions agree). Torque is a vector; in two dimensions treat anticlockwise as positive and clockwise as negative. A couple is a pair of equal and opposite forces whose lines of action do not coincide. A couple produces no net force, so it cannot change the translational motion, but it produces a torque equal to one of the forces multiplied by the perpendicular separation of the two lines of action, and this torque is the same about any axis. The definition of a couple is not in the booklet.

2. Equilibrium and the centre of mass

A rigid body is in equilibrium only when two conditions hold at once: translational equilibrium, the vector sum of the forces is zero, and rotational equilibrium, the sum of the torques about any axis is zero. Both conditions must be stated and used; neither is in the booklet. Because the torque sum is zero about every axis in equilibrium, you may choose the axis to suit yourself: put it through the point where an unknown force acts so that force drops out of the torque equation. The centre of mass is the point at which the whole mass may be taken to act for the purpose of calculating the torque of gravity; for a uniform symmetric body it lies at the geometric centre. A ladder, a beam on supports or a hanging sign is solved by writing one torque equation about a well chosen axis and, if needed, the force balance in each direction.

3. Rotational kinematics and the link to linear motion

Angular displacement θ is measured in radians, angular velocity ω in rad s^{-1} and angular acceleration α in rad s^{-2}. One revolution is 2π rad. For constant angular acceleration the four rotational SUVAT equations are in the booklet: $\omega_f = \omega_i + \alpha t$, $\Delta\theta = \frac{(\omega_f+\omega_i)t}{2}$, $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$ and $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$. They are used exactly like the linear versions: list the known quantities, pick the equation that contains the unknown, and convert revolutions to radians before substituting. A point at distance r from the axis has linear speed $v = \omega r$ and tangential acceleration $a = \alpha r$; $v = \omega r$ is in the booklet, but only under circular motion (A.2); $a = \alpha r$ is not in the booklet and must be learned. A graph of ω against t has gradient α and the area under it is $\Delta\theta$.

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4. Moment of inertia and Newton's second law for rotation

The moment of inertia of a system of point masses about an axis is $I = \sum mr^2$, where each r is the perpendicular distance of that mass from the axis. It plays the role of mass in rotation: it measures how hard it is to change the angular velocity. Because r is squared, mass far from the axis contributes much more than mass near it, and the same body has different moments of inertia about different axes. For extended bodies (a solid disc or cylinder, a hoop, a sphere, a rod) the formula will be given in the question; do not quote one from memory and do not confuse a hoop or a mass on the rim ($I = mr^2$) with a solid disc (where the given formula contains a factor of one half). Newton's second law for rotation is $\tau = I\alpha$ (in the booklet), where τ is the net torque about the axis. A typical flywheel or pulley question gives a force applied tangentially at the rim: compute $\tau = Fr$, then $\alpha = \tau/I$, then use rotational SUVAT.

5. Angular momentum, angular impulse and conservation

Angular momentum is $L = I\omega$, unit kg m^2 s^{-1}. Angular impulse is the change in angular momentum produced by a torque acting for a time: $\Delta L = \tau\Delta t = \Delta(I\omega)$ (in the booklet), with unit N m s, which is the same as kg m^2 s^{-1}. Both forms are needed: a torque graph against time has area equal to the angular impulse. If no external torque acts on a system, its total angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$. The rotational analogue of an isolated system is one with no net external torque, so internal forces (a skater pulling in her arms, a star contracting under its own gravity, putty sticking to a rotating disc) do not change L. When I decreases, ω increases in inverse proportion ($\omega \propto 1/I$ at constant L): halving I doubles ω. Kinetic energy is generally not conserved in these events: the skater does work with her muscles, so her kinetic energy rises; in a sticking collision between rotating bodies the frictional forces that bring them to a common angular velocity dissipate energy, so kinetic energy falls. The useful form $E_k = \frac{L^2}{2I}$ (in the booklet) makes this explicit: with L fixed, a smaller I means a larger kinetic energy.

6. Rotational kinetic energy and rolling without slipping

A rotating body has kinetic energy $E_k = \frac{1}{2}I\omega^2$ (in the booklet). A body that rolls without slipping is doing two things at once: its centre of mass translates with speed v and the body rotates about the centre of mass with angular velocity ω, so its total kinetic energy is $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. The rolling condition $v = \omega r$ (not stated as a rolling condition in the booklet, although $v = \omega r$ appears under A.2) links the two parts. Substituting $I = kmr^2$, where k is the numerical factor supplied in the question, gives $E_{k,\text{rot}} = \frac{1}{2}kmv^2$, so the ratio of rotational to translational kinetic energy is simply k and does not depend on mass, radius or speed. On an incline with no slipping, friction does no work, so $mgh = \frac{1}{2}mv^2(1+k)$ and $v^2 = \frac{2gh}{1+k}$. A body with larger k (a hoop, $k = 1$) reaches the bottom more slowly than one with smaller k (a solid sphere), whatever their masses and radii. If only part of a system rotates (a cart whose wheels roll while its body only translates), the translational term uses the total mass while the rotational term uses only the rotating parts, so the ratio is no longer simply k.

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Formulas

$\tau = Fr\sin\theta$	in the booklet
$\omega_f = \omega_i + \alpha t,\ \Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2,\ \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta,\ \Delta\theta = \frac{(\omega_f+\omega_i)t}{2}$	in the booklet
$I = \sum mr^2$	in the booklet
$\tau = I\alpha$	in the booklet
$L = I\omega,\ \Delta L = \tau\Delta t = \Delta(I\omega)$	in the booklet
$E_k = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$	in the booklet
$v = \omega r,\ a = \alpha r$	learn it (booklet: only in A.2)
Equilibrium: $\sum F = 0$ and $\sum \tau = 0$ about any axis	learn it
Couple: $\tau = Fd$ with d the perpendicular separation; net force zero	learn it
Rolling without slipping: $E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ with $v = \omega r$	learn it

Worked examples

Example 1. A uniform solid cylinder of mass M and radius R is released from rest at the top of an incline of vertical height $h = 1.2$ m and rolls without slipping to the bottom. The moment of inertia of the cylinder about its axis is $I = \frac{1}{2}MR^2$. (a) Show that the ratio of rotational kinetic energy to translational kinetic energy is $\frac{1}{2}$. (b) Determine the speed of the centre of the cylinder at the bottom. (c) Explain why the result does not depend on M or R.

1. (a) Rolling without slipping gives $\omega = v/R$. Rotational kinetic energy: $\frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{1}{2}MR^2\times\frac{v^2}{R^2} = \frac{1}{4}Mv^2$.
2. Translational kinetic energy is $\frac{1}{2}Mv^2$, so the ratio is $\frac{1}{4}Mv^2 \div \frac{1}{2}Mv^2 = \frac{1}{2}$.
3. (b) No slipping means friction does no work, so $Mgh = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2$.
4. $v^2 = \frac{4gh}{3} = \frac{4\times9.8\times1.2}{3} = 15.7\ \text{m}^2\ \text{s}^{-2}$, so $v = 3.96\ \text{m s}^{-1}$.
5. (c) M cancels from every term of the energy equation, and R cancels because $I \propto R^2$ while $\omega^2 \propto 1/R^2$; only the shape factor $\frac{1}{2}$ in I and the height h remain.

Answer: (a) Ratio $= \frac{1}{2}$. (b) $v \approx 4.0\ \text{m s}^{-1}$. (c) Both M and R cancel; the speed depends only on h and the shape factor in I.

Example 2. A flywheel is a uniform disc of mass 4.0 kg and radius 0.20 m, free to rotate about a frictionless horizontal axle through its centre. The moment of inertia of a uniform disc is $I = \frac{1}{2}MR^2$. A string wrapped around the rim is pulled with a constant tension of 12 N for 5.0 s, starting from rest. (a) Calculate the angular acceleration of the flywheel. (b) Determine the number of revolutions made in the 5.0 s. (c) State the angular momentum of the flywheel after 5.0 s.

1. (a) $I = \frac{1}{2}\times 4.0\times 0.20^2 = 0.080\ \text{kg m}^2$. The tension acts tangentially at the rim, so $\theta = 90^\circ$ and $\tau = FR = 12\times0.20 = 2.4\ \text{N m}$.
2. $\alpha = \tau/I = 2.4/0.080 = 30\ \text{rad s}^{-2}$.
3. (b) From rest, $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}\times30\times5.0^2 = 375$ rad.

4. Number of revolutions $= 375/2\pi = 59.7 \approx 60$.

5. (c) $\omega_f = \alpha t = 30 \times 5.0 = 150$ rad s^{-1}, so $L = I\omega = 0.080 \times 150 = 12$ kg m^2 s^{-1}. Check: angular impulse $\tau\Delta t = 2.4 \times 5.0 = 12$ N m s, the same value.

Answer: (a) $\alpha = 30$ rad s^{-2}. (b) About 60 revolutions. (c) $L = 12$ kg m^2 s^{-1}.

Example 3. A horizontal turntable of moment of inertia 0.60 kg m^2 rotates freely at 4.0 rad s^{-1} about a vertical axis through its centre. A lump of putty of mass 0.50 kg is dropped vertically onto the turntable and sticks at a distance of 0.40 m from the axis. (a) Determine the angular velocity of the turntable after the putty sticks. (b) Calculate the kinetic energy of the system before and after. (c) Explain why the kinetic energy changes even though angular momentum is conserved.

1. (a) The putty falls vertically, so it carries no angular momentum about the axis and no external torque acts about the axis; angular momentum is conserved.

2. Treat the putty as a point mass: $I_{\text{putty}} = mr^2 = 0.50 \times 0.40^2 = 0.080$ kg m^2. Total after: $I_2 = 0.60 + 0.080 = 0.68$ kg m^2.

3. $I_1\omega_1 = I_2\omega_2 \Rightarrow \omega_2 = \frac{0.60 \times 4.0}{0.68} = 3.5$ rad s^{-1}.

4. (b) Before: $E_k = \frac{1}{2} \times 0.60 \times 4.0^2 = 4.8$ J. After: $E_k = \frac{1}{2} \times 0.68 \times 3.53^2 = 4.2$ J. (Equivalently $E_k = L^2/2I$ with $L = 2.4$ kg m^2 s^{-1} fixed and I increased.)

5. (c) The putty must be accelerated from rest to the speed of the turntable by friction between putty and surface; this is an inelastic collision in which friction does negative net work on the system, converting about 0.6 J of kinetic energy into thermal energy. Friction here is an internal force, so it changes kinetic energy but not angular momentum.

Answer: (a) $\omega_2 \approx 3.5$ rad s^{-1}. (b) 4.8 J before, 4.2 J after. (c) Internal friction during the sticking collision dissipates kinetic energy; it exerts no external torque, so L is unchanged.

Common errors

- Using the full distance r instead of the perpendicular distance when the force is not at right angles to the lever arm. Always draw the line of action of the force and use $r \sin\theta$, or resolve the force into the component perpendicular to the arm.
- Applying the solid disc formula to a mass on the rim, or vice versa. A point mass or thin hoop at radius r has $I = mr^2$; a solid disc has the factor given in the question. Read the stem and check which object is rotating.
- Treating rolling as pure translation, so the gravitational potential energy all becomes $\frac{1}{2}mv^2$ and the speed comes out too large. Write both kinetic energy terms and substitute $\omega = v/r$ before solving.
- Claiming angular momentum is conserved while an external torque acts, for example a wheel being braked or a pulley driven by a hanging mass. Conservation applies only when the net external torque on the chosen system is zero; otherwise use $\Delta L = \tau\Delta t$.
- Forgetting the half in $\Delta\theta = \frac{1}{2}\alpha t^2$ when starting from rest, or leaving the answer in radians when the question asks for revolutions. Divide by 2π at the end.
- Giving the unit of angular impulse as N m or as kg m^2 s^{-2}. Angular impulse equals torque multiplied by time, N m s, identical to the unit of angular momentum, kg m^2 s^{-1}.

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Before the exam, can you...

- Before the exam, can you calculate a torque from a force, a distance and an angle, and explain why a couple gives a torque but no net force?
- Can you write both conditions of equilibrium and choose an axis that removes an unknown force from the torque equation?
- Can you solve a constant angular acceleration problem with the rotational SUVAT equations, converting between radians, revolutions and linear speed using $v = \omega r$?
- Can you compute $I = \sum mr^2$ for point masses, apply $\tau = I\alpha$ to a flywheel or pulley, and find the angular momentum and angular impulse?
- Can you set up an energy equation for a rolling body on an incline with $v = \omega r$, and explain qualitatively why kinetic energy changes when a skater or disc changes its moment of inertia at constant L?

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