Limits, continuity and the derivative from first principles: practice worksheet

IB Mathematics: analysis and approaches HL · SL 5.1, AHL 5.12-5.13 · 52 minutes · 43 marks

Question 1 (*no calculator*)

The function f is defined by

$$f(x) = \begin{cases} 3 - x, & x \leq 2, \\ x - 1, & x > 2. \end{cases}$$

(a) Find $\lim_{x \to 2} f(x)$. [1]

(b) Explain why f is continuous but not differentiable at $x = 2$. [2]

Question 2 (*no calculator*)

Let $f(x) = 3x^2 - 4x + 7$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$ from first principles. [4]

Question 3 (*no calculator*)

The curve C has equation $y = f(x)$, where $f(x) = \dfrac{5}{x+2}$ and $x \neq -2$.

(a) Show from first principles that $f'(x) = -\dfrac{5}{(x+2)^2}$. [4]

(b) Hence find the equation of the tangent to C at $x = -1$. [2]

Question 4 (*no calculator*)

Let $g(x) = e^{2x} - 1 - 2x$.

(a) Write down $g''(x)$ and $g'''(x)$. [2]

(b) Use l'Hôpital's rule to determine $\lim_{x \to 0} \dfrac{g(x)}{x^2}$. State the indeterminate form before each application. [5]

Question 5 (*no calculator*)

Let a be a positive real number, and define

$$L(a) = \lim_{x \to 0} \frac{\sin(ax) - ax}{x - \sin x}.$$

Angles are measured in radians.

(a) By applying l'Hôpital's rule, show that $L(a) = -a^3$. State the indeterminate form before each application. [7]

(b) Hence determine a when $L(a) = -8$. [2]

Question 6 (*no calculator*)

Let $p > 0$. For $x > -\frac{1}{p}$, the function F is defined by

$$F(x) = \begin{cases} \dfrac{\ln(1 + px)}{x}, & x \neq 0, \\ c, & x = 0. \end{cases}$$

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(a) Use l'Hôpital's rule to determine c, in terms of p, so that F is continuous at $x = 0$. [3]

(b) Using the definition of the derivative from first principles and l'Hôpital's rule, show that $F'(0) = -\frac{p^2}{2}$. [6]

(c) You may use the Maclaurin expansion

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots,$$

valid for $|u| < 1$. Use this expansion to find $\lim_{x\to 0} \dfrac{F(x) - p + \frac{p^2}{2}x}{x^2}$. [2]

(d) The tangent to the curve $y = F(x)$ at $x = 0$ passes through $(1, -4)$. Determine p and find the equation of this tangent. [3]

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Markscheme

Question 1

(a)

$\lim_{x\to 2} f(x) = 1.$ **A1**

(b)

The limit equals $f(2) = 1$, establishing continuity. **R1**
The left-hand gradient is -1 and the right-hand gradient is 1; their inequality establishes that f is not differentiable at $x = 2$. **R1**

Accept an explanation identifying a corner and its two different gradients.

Question 2

(a)

Form $f'(x) = \lim_{h\to 0} \dfrac{3(x+h)^2 - 4(x+h) + 7 - (3x^2 - 4x + 7)}{h}$. **M1**
Expand and collect the numerator to obtain $6xh + 3h^2 - 4h$. **A1**
Simplify the difference quotient to $6x + 3h - 4$ for $h \neq 0$. **A1**
Take the limit explicitly: $f'(x) = \lim_{h\to 0}(6x + 3h - 4) = 6x - 4$. **A1**

A derivative obtained solely by a differentiation rule earns no marks.

Question 3

(a)

Form $f'(x) = \lim_{h\to 0} \dfrac{\frac{5}{x+h+2} - \frac{5}{x+2}}{h}$. **M1**
Combine the fractions to obtain $\dfrac{-5h}{h(x+h+2)(x+2)}$. **A1**
Cancel h to obtain $-\dfrac{5}{(x+h+2)(x+2)}$. **A1**
Take the limit explicitly to obtain $-\dfrac{5}{(x+2)^2}$. **A1**
$f'(x) = -\dfrac{5}{(x+2)^2}$. **AG**

The difference quotient is considered for $h \neq 0$ and $x + h \neq -2$.

(b)

Obtain the tangent gradient $f'(-1) = -5$. **A1**
Use $f(-1) = 5$ to obtain $y - 5 = -5(x + 1)$, or $y = -5x$. **A1**

Accept equivalent equations. Allow FT for the tangent equation using their finite gradient from part (a) and the correct point $(-1, 5)$.

Question 4

(a)

$g''(x) = 4e^{2x}$. **A1**
$g'''(x) = 8e^{2x}$. **A1**

Accept equivalent notation for the second and third derivatives.

(b)

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State that the original quotient has the indeterminate form $0/0$. **R1**

Obtain $\lim_{x\to 0}\dfrac{2e^{2x}-2}{2x}$. **A1**

State that the new quotient still has the indeterminate form $0/0$. **R1**

Apply l'Hôpital's rule again to obtain $\lim_{x\to 0}\dfrac{4e^{2x}}{2}$. **M1**

Evaluate the limit as 2. **A1**

Award differentiation marks even if the limit symbol is omitted in intermediate working. The prescribed method must be used.

Question 5

(a)

State that the original quotient has the indeterminate form $0/0$. **R1**

Obtain $\lim_{x\to 0}\dfrac{a\cos(ax)-a}{1-\cos x}$. **A1**

State that this quotient has the indeterminate form $0/0$. **R1**

Apply l'Hôpital's rule again to obtain $\lim_{x\to 0}\dfrac{-a^2\sin(ax)}{\sin x}$. **M1**

State that this quotient still has the indeterminate form $0/0$. **R1**

Apply l'Hôpital's rule a third time to obtain $\lim_{x\to 0}\dfrac{-a^3\cos(ax)}{\cos x}$. **M1**

Substitute $x=0$ to obtain $-a^3$. **A1**

$L(a)=-a^3$. **AG**

The final quotient is no longer indeterminate, so no further application is required.

(b)

Set $-a^3=-8$, giving $a^3=8$. **M1**

$a=2$. **A1**

Question 6

(a)

Identify the indeterminate form $0/0$ in $\lim_{x\to 0}\dfrac{\ln(1+px)}{x}$. **R1**

Apply l'Hôpital's rule to obtain $\lim_{x\to 0}\dfrac{p}{1+px}$. **M1**

The limit is p, so continuity requires $c=p$. **A1**

Use $c=p$ in the remaining parts.

(b)

Form $F'(0)=\lim_{h\to 0}\dfrac{F(h)-F(0)}{h}=\lim_{h\to 0}\dfrac{\ln(1+ph)-ph}{h^2}$. **M1**

State that the resulting quotient has the indeterminate form $0/0$. **R1**

Obtain $\lim_{h\to 0}\dfrac{\frac{p}{1+ph}-p}{2h}$. **A1**

State that this quotient still has the indeterminate form $0/0$. **R1**

Apply l'Hôpital's rule again to obtain $\lim_{h\to 0}\dfrac{-p^2/(1+ph)^2}{2}$. **M1**

Evaluate the limit as $-\dfrac{p^2}{2}$. **A1**

$F'(0)=-\dfrac{p^2}{2}$. **AG**

Accept algebraic simplification of the first differentiated quotient before taking its limit. The first-principles

difference quotient must be shown.

(c)

Obtain $F(x) = p - \frac{p^2}{2}x + \frac{p^3}{3}x^2 - \frac{p^4}{4}x^3 + \cdots$ near $x = 0$. **A1**

Cancel the constant and linear terms, divide by x^2, and obtain the limit $\frac{p^3}{3}$. **A1**

It is sufficient to show the expansion through the quadratic term with an indication that further terms follow.

(d)

Use the tangent equation $y = p - \frac{p^2}{2}x$ and the given point to obtain $-4 = p - \frac{p^2}{2}$. **M1**

Solve $(p-4)(p+2) = 0$ and select $p = 4$ since $p > 0$. **A1**

Obtain the tangent equation $y = 4 - 8x$. **A1**

Accept equivalent tangent equations. Allow FT for the final equation using their positive value of p and their gradient from part (b).

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