

Integration by substitution and by parts: practice worksheet

IB Mathematics: analysis and approaches HL · AHL 5.16 · 50 minutes · 42 marks

Question 1 (no calculator)

- (a) Find $\int x \cos 2x \, dx$. [3]

Question 2 (no calculator)

- (a) Using the substitution $u = 1 + x^2$, find the exact value of $\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}} \, dx$. [4]

Question 3 (no calculator)

- (a) Find $\int x^2 \sin x \, dx$. [5]
 (b) Hence show that $\int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4$. [2]

Question 4 (no calculator)

- (a) Express $\frac{1}{u(u+1)}$ in partial fractions. [2]
 (b) Hence, using the substitution $u = \cos x$, find the exact value of $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x + \cos x} \, dx$. [5]

Question 5 (no calculator)

Let $I = \int e^{2x} \cos x \, dx$.

- (a) Using integration by parts twice, show that $I = \frac{1}{5}e^{2x} (\sin x + 2 \cos x) + c$. [6]
 (b) The region R is bounded by the curve $y = e^{2x} \cos x$, the x -axis and the y -axis, for $0 \leq x \leq \frac{\pi}{2}$. Find the exact area of R . [2]

Question 6 (no calculator)

A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} kxe^x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a positive constant.

- (a) Using integration by parts, show that $k = 1$. [4]
 (b) Find $P(X > \ln 2)$, giving your answer in the form $a + b \ln 2$ where $a, b \in \mathbb{Z}$. [3]
 (c) Show that $E(X) = e - 2$. [4]
 (d) The median of X is m . Show that m satisfies the equation $(1 - m)e^m = \frac{1}{2}$. [2]

Markscheme

Question 1

(a)

attempt at integration by parts with $u = x$, $\frac{dv}{dx} = \cos 2x$ **M1**

$$\frac{x}{2} \sin 2x - \int \frac{1}{2} \sin 2x \, dx$$
 A1

$$\frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + c$$
 A1

Award M1 for a correct structure $uv - \int v u'$ even with a sign or coefficient error in v . Condone absence of $+c$ until the final A1.

Question 2

(a)

$$\frac{du}{dx} = 2x \text{ (or } du = 2x \, dx)$$
 A1

$$\frac{1}{2} \int u^{-\frac{1}{2}} \, du$$
 M1

new limits $u = 1$ and $u = 4$ (or returning to x and using original limits) **A1**

$$[\sqrt{u}]_1^4 = 2 - 1 = 1$$
 A1

Condone absence of or incorrect limits up to the M1. The third A1 may be earned by writing $\sqrt{1+x^2}$ and substituting $x = 0$, $x = \sqrt{3}$.

Question 3

(a)

attempt at integration by parts with $u = x^2$, $\frac{dv}{dx} = \sin x$ **M1**

$$-x^2 \cos x + \int 2x \cos x \, dx$$
 A1

second application of parts with $u = 2x$ (or x), $\frac{dv}{dx} = \cos x$ **M1**

$$2x \sin x - \int 2 \sin x \, dx$$
 A1

$$-x^2 \cos x + 2x \sin x + 2 \cos x + c$$
 A1

Award the first A1 for both the correct uv term and the correct remaining integral; if only one is correct award M1A0. Second M1 requires the correct direction of differentiation (polynomial degree reduced).

(b)

$$\text{substituting limits into their answer to (a): } (-\pi^2(-1) + 0 + 2(-1)) - (0 + 0 + 2)$$
 M1

$$= (\pi^2 - 2) - 2 = \pi^2 - 4$$
 A1

$$\pi^2 - 4$$
 AG

The A1 requires the values $\cos \pi = -1$, $\sin \pi = 0$, $\cos 0 = 1$ seen or clearly used. Do not award A1 if the answer to (a) is incorrect and the given result is reached by manipulation.

Question 4

(a)

$$\frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{u+1} \text{ and attempt to find } A, B \text{ (e.g. } 1 = A(u+1) + Bu)$$
 M1

$$\frac{1}{u} - \frac{1}{u+1}$$
 A1

Accept the cover-up method with $A = 1$, $B = -1$ stated.

(b)

$$\begin{aligned}\frac{du}{dx} &= -\sin x \text{ (or } du = -\sin x \, dx) & \text{A1} \\ -\int \frac{1}{u(u+1)} \, du &= -\int \left(\frac{1}{u} - \frac{1}{u+1} \right) du & \text{M1} \\ \text{limits } u = 1 \text{ and } u = \frac{1}{2} \text{ (or equivalent, e.g. } \int_{\frac{1}{2}}^1 \text{ with sign reversed)} & & \text{A1} \\ [-\ln u + \ln(u+1)]_{\frac{1}{2}}^1 \text{ or } \left[\ln \frac{u+1}{u} \right]_{\frac{1}{2}}^1 & & \text{A1} \\ \ln 3 - \ln 2 = \ln \frac{3}{2} & & \text{A1}\end{aligned}$$

Condone absence of or incorrect limits up to the second A1. Condone missing modulus signs. Accept $\ln 3 - \ln 2$ or $-\ln \frac{2}{3}$ for the final A1.

Question 5

(a)

$$\begin{aligned}\text{attempt at integration by parts with } u = e^{2x}, \frac{dv}{dx} &= \cos x & \text{M1} \\ I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx & & \text{A1} \\ \text{second application of parts on } \int e^{2x} \sin x \, dx \text{ with } u = e^{2x}, \frac{dv}{dx} &= \sin x & \text{M1} \\ \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx & & \text{A1} \\ I = e^{2x} \sin x + 2e^{2x} \cos x - 4I \text{ (recognising the original integral)} & & \text{M1} \\ 5I = e^{2x}(\sin x + 2 \cos x) + k \text{ so } I = \frac{1}{5}e^{2x}(\sin x + 2 \cos x) + c & & \text{A1} \\ I = \frac{1}{5}e^{2x}(\sin x + 2 \cos x) + c & & \text{AG}\end{aligned}$$

METHOD 2: taking $u = \cos x$, $\frac{dv}{dx} = e^{2x}$ both times gives $I = \frac{1}{2}e^{2x} \cos x + \frac{1}{4}e^{2x} \sin x - \frac{1}{4}I$; award marks equivalently. Candidates must use the same pairing in both applications, otherwise the identity $I = I$ results and at most M1A1M1A0 is available.

(b)

$$\begin{aligned}\text{area} &= \left[\frac{1}{5}e^{2x}(\sin x + 2 \cos x) \right]_0^{\frac{\pi}{2}} & \text{M1} \\ &= \frac{1}{5}(e^{\pi}(1+0) - (0+2)) = \frac{e^{\pi}-2}{5} & \text{A1}\end{aligned}$$

Accept $\frac{1}{5}e^{\pi} - \frac{2}{5}$. Since the curve is non-negative on the interval no modulus is needed.

Question 6

(a)

$$\begin{aligned}\text{attempt at integration by parts on } \int xe^x \, dx \text{ with } u = x, \frac{dv}{dx} &= e^x & \text{M1} \\ \int xe^x \, dx = xe^x - e^x (= (x-1)e^x) & & \text{A1} \\ k \int_0^1 xe^x \, dx = 1, \text{ i.e. } k[(x-1)e^x]_0^1 = 1 & & \text{M1} \\ k(0 - (-1)) = 1 \Rightarrow k = 1 & & \text{A1} \\ k = 1 & & \text{AG}\end{aligned}$$

The second M1 requires the total probability condition with limits 0 and 1 to be explicitly stated. The final A1 requires the values at both limits to be shown.

(b)

$$\begin{aligned}P(X > \ln 2) &= \int_{\ln 2}^1 xe^x \, dx \text{ or } 1 - \int_0^{\ln 2} xe^x \, dx & \text{M1} \\ \text{use of } e^{\ln 2} = 2: [(x-1)e^x]_{\ln 2}^1 = 0 - 2(\ln 2 - 1) & & \text{A1} \\ = 2 - 2 \ln 2 & & \text{A1}\end{aligned}$$

Award the first A1 for $(\ln 2 - 1) \cdot 2 + 1 = 2 \ln 2 - 1$ if the complement route is used. Accept $2(1 - \ln 2)$.

(c)

$$E(X) = \int_0^1 x^2 e^x dx \quad \text{M1}$$

attempt at integration by parts with $u = x^2$, $\frac{dv}{dx} = e^x$ M1

$$x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2(x-1)e^x \text{ (using part (a))} \quad \text{A1}$$

$$[x^2 e^x - 2(x-1)e^x]_0^1 = (e-0) - (0+2) = e-2 \quad \text{A1}$$

$$E(X) = e-2 \quad \text{AG}$$

Accept a full second application of parts on $\int x e^x dx$ in place of quoting part (a). Accept the antiderivative written as $e^x(x^2 - 2x + 2)$.

(d)

$$\int_0^m x e^x dx = \frac{1}{2}, \text{ i.e. } [(x-1)e^x]_0^m = \frac{1}{2} \quad \text{M1}$$

$$(m-1)e^m + 1 = \frac{1}{2} \Rightarrow (1-m)e^m = \frac{1}{2} \quad \text{A1}$$

$$(1-m)e^m = \frac{1}{2} \quad \text{AG}$$

Accept the equivalent condition $\int_m^1 x e^x dx = \frac{1}{2}$ giving $-(m-1)e^m = \frac{1}{2}$ directly.