Integration: antiderivatives and standard integrals: practice worksheet

IB Mathematics: analysis and approaches HL · SL 5.5, 5.10, AHL 5.15 · 48 minutes · 40 marks

Question 1 (*no calculator*)

No calculator. Let $x > 0$.

(a) Find $\int \left(6x^{1/2} - 4x^{-2}\right) \mathrm{d}x$. [3]

Question 2 (*no calculator*)

No calculator. A function f satisfies $f'(x) = 3\cos(3x) - 2\mathrm{e}^{-2x}$ and $f(0) = 4$. Angles are measured in radians.

(a) Find $f(x)$. [4]

Question 3 (*no calculator*)

No calculator. All functions in this question are defined for real x.

(a) Find $\int \dfrac{2x}{x^2+3} \mathrm{d}x$. [2]

(b) A function h satisfies $h'(x) = \frac{2x}{x^2+3} + 6x(x^2+3)^2$ and $h(1) = 0$. Hence find $h(x)$. [4]

Question 4 (*no calculator*)

No calculator. Let $x > 1$.

(a) Show that $\dfrac{5x+1}{(x-1)(x+2)} = \dfrac{2}{x-1} + \dfrac{3}{x+2}$. [3]

(b) A function F satisfies $F'(x) = \frac{5x+1}{(x-1)(x+2)}$ and $F(2) = 0$. Hence find $F(x)$. [4]

Question 5 (*no calculator*)

No calculator. Give each answer in terms of an inverse trigonometric function.

(a) Find $\int \dfrac{1}{x^2 - 4x + 13} \mathrm{d}x$. [4]

(b) For $-2 < x < 6$, find $\int \dfrac{1}{\sqrt{12 + 4x - x^2}} \mathrm{d}x$. [4]

Question 6 (*no calculator*)

No calculator. A differentiable function f, defined for all real x, satisfies $f'(x) = \dfrac{2x(x^2-2)}{(x^2+1)(x^2+4)}$ and $f(0) = 0$.

(a) For $u \geq 0$, show that $\dfrac{u-2}{(u+1)(u+4)} = -\dfrac{1}{u+1} + \dfrac{2}{u+4}$. [2]

(b) By using the substitution $u = x^2$ and the result of part (a), find the general expression for $f(x)$, including a constant of integration. [4]

(c) Hence determine $f(x)$, expressing your answer as a single logarithm. [2]

(d) Find the coordinates of all stationary points of the graph of f and determine their nature. Justify your classifications using the sign of $f'(x)$. [4]

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Markscheme

Question 1

(a)

Attempt to use $\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$ on at least one term.	**M1**
Obtain $4x^{3/2} + 4x^{-1}$.	**A1**
Include an arbitrary constant $+C$.	**A1**

Accept equivalent forms.

Question 2

(a)

Attempt antiderivatives involving $\sin(3x)$ and e^{-2x}.	**M1**
Obtain $f(x) = \sin(3x) + \mathrm{e}^{-2x} + C$.	**A1**
Substitute the condition to obtain $4 = 0 + 1 + C$.	**M1**
Obtain $C = 3$ and hence $f(x) = \sin(3x) + \mathrm{e}^{-2x} + 3$.	**A1**

The final A1 requires a constant to have been introduced and determined. FT for the boundary-condition method using their integrated expression.

Question 3

(a)

Recognise the reverse chain rule, or use $u = x^2 + 3$ with $\mathrm{d}u = 2x \,\mathrm{d}x$.	**M1**
Obtain $\ln(x^2 + 3) + C$.	**A1**

Accept $\ln|x^2 + 3| + C$*; the modulus is unnecessary.*

(b)

Use part (a) and attempt reverse chain rule for $6x(x^2 + 3)^2$.	**M1**
Obtain $h(x) = \ln(x^2 + 3) + (x^2 + 3)^3 + C$.	**A1**
Use $h(1) = 0$ to obtain $0 = \ln 4 + 64 + C$.	**M1**
Obtain $h(x) = \ln\left(\frac{x^2+3}{4}\right) + (x^2 + 3)^3 - 64$.	**A1**

Accept the unsimplified form with $C = -\ln 4 - 64$*. FT for the boundary-condition method using their antiderivative. The final A1 requires the constant to have been determined.*

Question 4

(a)

Write $5x + 1 = A(x + 2) + B(x - 1)$, or combine the given fractions over a common denominator.	**M1**
Obtain $A + B = 5$ and $2A - B = 1$, or the numerator $2(x + 2) + 3(x - 1)$.	**A1**
Obtain $A = 2$, $B = 3$, or simplify the combined numerator to $5x + 1$.	**A1**
Hence $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$.	**AG**

A convincing algebraic verification is required.

(b)

Attempt to integrate the two partial fractions separately.	**M1**
Obtain $F(x) = 2\ln(x - 1) + 3\ln(x + 2) + C$.	**A1**
Use $F(2) = 0$ to obtain $0 = 3\ln 4 + C$.	**M1**
Obtain $F(x) = 2\ln(x - 1) + 3\ln\left(\frac{x+2}{4}\right)$.	**A1**

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Accept equivalent logarithmic forms and modulus signs. FT for the boundary-condition method using their integrated expression. The final A1 requires the constant to have been determined.

Question 5

(a)

Attempt to complete the square in the denominator.	**M1**
Obtain $x^2 - 4x + 13 = (x-2)^2 + 9$.	**A1**
Use $u = \frac{x-2}{3}$ to obtain $\frac{1}{3}\int \frac{1}{1+u^2}\,\mathrm{d}u$, or attempt the corresponding standard arctangent form.	**M1**
Obtain $\frac{1}{3}\arctan\left(\frac{x-2}{3}\right) + C$.	**A1**

Accept equivalent inverse-tangent notation.

(b)

Rewrite the expression under the square root as $16 - (x-2)^2$.	**M1**
Use $u = \frac{x-2}{4}$ to obtain $\int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u$, or attempt the corresponding standard arcsine form.	**M1**
Obtain the antiderivative $\arcsin u$ in the substituted variable, or $\arcsin\left(\frac{x-2}{4}\right)$ directly.	**A1**
State $\arcsin\left(\frac{x-2}{4}\right) + C$.	**A1**

Accept $-\arccos\left(\frac{x-2}{4}\right) + C$.

Question 6

(a)

Combine the right-hand side to obtain $\frac{-(u+4)+2(u+1)}{(u+1)(u+4)}$.	**M1**
Simplify the numerator to $u - 2$.	**A1**
Hence $\frac{u-2}{(u+1)(u+4)} = -\frac{1}{u+1} + \frac{2}{u+4}$.	**AG**

Accept a partial-fraction method with coefficients determined explicitly.

(b)

Use $u = x^2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$.	**M1**
Obtain $f(x) = \int \left(-\frac{1}{u+1} + \frac{2}{u+4}\right)\mathrm{d}u$.	**A1**
Integrate to obtain $-\ln(u+1) + 2\ln(u+4)$.	**A1**
Substitute back and include the constant: $f(x) = -\ln(x^2+1) + 2\ln(x^2+4) + C$.	**A1**

Accept equivalent logarithmic forms. Modulus signs are unnecessary but may be included.

(c)

Use $f(0) = 0$ to obtain $0 = 2\ln 4 + C$, giving $C = -\ln 16$.	**M1**
Obtain $f(x) = \ln\left(\frac{(x^2+4)^2}{16(x^2+1)}\right)$.	**A1**

FT using their expression from part (b), provided a constant is determined and logarithm laws are applied correctly.

(d)

Set $f'(x) = 0$ and solve $2x(x^2-2) = 0$, using the fact that the denominator is positive.	**M1**
Obtain $x = -\sqrt{2}$, 0 and $\sqrt{2}$.	**A1**
Obtain all coordinates: $\left(-\sqrt{2}, \ln\frac{3}{4}\right)$, $(0,0)$ and $\left(\sqrt{2}, \ln\frac{3}{4}\right)$.	**A1**
Establish the sign sequence negative, positive, negative, positive, and hence classify $x = \pm\sqrt{2}$ as local minima and $x = 0$ as a local maximum.	**R1**

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FT for ordinates calculated consistently from their function in part (c). *Accept equivalent exact logarithmic values. Classifications require supporting sign information.*

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