

Implicit differentiation, related rates and optimisation: practice worksheet

IB Mathematics: analysis and approaches HL · AHL 5.14 · 55 minutes · 44 marks

Question 1 (no calculator)

The curve C has equation $x^2 + xy + y^2 = 7$.

Find the gradient of C at the point $(1, 2)$. [3]

Question 2 (no calculator)

The edge length x cm of a cube is increasing at a constant rate of 0.05 cm s^{-1} .

(a) Find the rate at which the volume of the cube is increasing at the instant when $x = 8$. [2]

(b) Find the rate at which the total surface area of the cube is increasing at the same instant. [2]

Question 3 (no calculator)

The curve C has equation $x^2 - xy + y^2 = 12$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{2y-x}$. [3]

(b) Find the coordinates of the points on C at which the tangent is horizontal. [3]

(c) Hence, or otherwise, find the coordinates of the points on C at which the tangent is vertical. [2]

Question 4 (calculator)

Sand falls from a conveyor belt at a constant rate of 0.6 m^3 per minute and forms a pile in the shape of a right circular cone. The height h metres of the pile is always three-quarters of its base radius r metres.

(a) Show that the volume $V \text{ m}^3$ of the pile is given by $V = \frac{16\pi h^3}{27}$. [2]

(b) Find the rate at which the height of the pile is increasing at the instant when $h = 1.5$. Give your answer in metres per minute correct to three significant figures. [4]

(c) Find the rate at which the base radius is increasing at the same instant. [1]

Question 5 (no calculator)

A farmer has 60 m of fencing and wishes to enclose a rectangular pen on the bank of a straight river. No fencing is needed along the river. The side of the pen parallel to the river has length x metres. Local regulations require that $x \geq 40$.

(a) Show that the area $A \text{ m}^2$ of the pen is given by $A = 30x - \frac{1}{2}x^2$, and state the domain of x . [2]

(b) Find $\frac{dA}{dx}$ and show that A has no stationary point within the domain. [3]

(c) Hence determine the maximum possible area of the pen, justifying your answer. [3]

Question 6 (calculator)

The number of bacteria in a colony, N thousand, after t hours satisfies the differential equation

$$\frac{dN}{dt} = 0.02N(60 - N), \quad 0 < N < 60.$$

Initially there are 10 thousand bacteria in the colony.

(a) By differentiating implicitly with respect to t , show that $\frac{d^2N}{dt^2} = 0.02(60 - 2N)\frac{dN}{dt}$. [3]

(b) Hence find the value of N for which the rate of growth of the colony is greatest, and justify that this value gives a maximum. [3]

- (c) Write down the maximum rate of growth of the colony. [1]
- (d) Solve the differential equation to show that $N = \frac{60}{1 + 5e^{-1.2t}}$. [5]
- (e) Find the time, in hours correct to three significant figures, at which the rate of growth of the colony is greatest. [2]

Markscheme

Question 1

attempts implicit differentiation, including product rule on xy and chain rule on y^2 **M1**

$$2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \text{ (or } \frac{dy}{dx} = -\frac{2x+y}{x+2y} \text{)} \quad \mathbf{A1}$$

$$\text{gradient} = -\frac{4}{5} \quad \mathbf{A1}$$

Award M1A0 if the chain rule factor $\frac{dy}{dx}$ is omitted from y^2 but present on xy . Substitution of $x = 1, y = 2$ before rearranging is acceptable.

Question 2

(a)

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \quad \mathbf{M1}$$

$$= 3(64)(0.05) = 9.6 \text{ (cm}^3 \text{ s}^{-1}\text{)} \quad \mathbf{A1}$$

Accept $\frac{48}{5}$.

(b)

$$S = 6x^2 \text{ so } \frac{dS}{dt} = 12x \frac{dx}{dt} \quad \mathbf{M1}$$

$$= 12(8)(0.05) = 4.8 \text{ (cm}^2 \text{ s}^{-1}\text{)} \quad \mathbf{A1}$$

Award M1A0 for $S = x^2$ leading to 0.8.

Question 3

(a)

attempts implicit differentiation with product rule on xy **M1**

$$2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad \mathbf{A1}$$

$$\frac{dy}{dx}(2y - x) = y - 2x \text{ leading to } \frac{dy}{dx} = \frac{y-2x}{2y-x} \quad \mathbf{A1}$$

AG

The rearrangement must be seen for the final A1.

(b)

sets numerator equal to zero: $y = 2x$ **M1**

substitutes into the equation of C: $x^2 - 2x^2 + 4x^2 = 12 \Rightarrow 3x^2 = 12$ **M1**

$(2, 4)$ and $(-2, -4)$ **A1**

Award A0 if only one point is given or if extra points are included.

(c)

sets denominator equal to zero, $x = 2y$, and substitutes: $4y^2 - 2y^2 + y^2 = 12$ **M1**

$(4, 2)$ and $(-4, -2)$ **A1**

Accept an argument from the symmetry of C in the line $y = x$ provided it is stated.

Question 4

(a)

$$r = \frac{4h}{3} \text{ substituted into } V = \frac{1}{3}\pi r^2 h \quad \mathbf{M1}$$

$$V = \frac{1}{3}\pi \cdot \frac{16h^2}{9} \cdot h = \frac{16\pi h^3}{27} \quad \mathbf{A1}$$

AG

(b)

$$\frac{dV}{dh} = \frac{16\pi h^2}{9} \quad \text{A1}$$

$$\text{uses } \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \quad \text{M1}$$

$$\frac{dV}{dh} = 4\pi \text{ when } h = 1.5 \text{ (seen anywhere)} \quad \text{A1}$$

$$\frac{dh}{dt} = \frac{0.6}{4\pi} = 0.0477 \text{ (m min}^{-1}\text{)} \quad \text{A1}$$

Accept $\frac{3}{20\pi}$. Award M1 for an equivalent chain rule statement such as $0.6 = \frac{16\pi h^2}{9} \frac{dh}{dt}$.

(c)

$$\frac{dr}{dt} = \frac{4}{3} \cdot \frac{dh}{dt} = 0.0637 \text{ (m min}^{-1}\text{)} \quad \text{A1}$$

FT from part (b). Accept $\frac{1}{5\pi}$.

Question 5

(a)

each side perpendicular to the river has length $\frac{60-x}{2}$, so $A = x \cdot \frac{60-x}{2} = 30x - \frac{1}{2}x^2$ M1

domain $40 \leq x < 60$ AG

Condone $40 \leq x \leq 60$. A1

(b)

$$\frac{dA}{dx} = 30 - x \quad \text{A1}$$

sets $\frac{dA}{dx} = 0$ to obtain $x = 30$ M1

$30 < 40$, so the stationary point is outside the domain R1

(c)

considers the sign of $\frac{dA}{dx}$ on the domain: $30 - x < 0$ for all $x \geq 40$ M1

A is decreasing throughout the domain, so the maximum occurs at the end point R1

$x = 40$

$$A = 30(40) - \frac{1}{2}(1600) = 400 \text{ (m}^2\text{)} \quad \text{A1}$$

Do not award R0A1. A sketch of the parabola with vertex at $x = 30$ and the domain marked is an acceptable justification.

Question 6

(a)

differentiates the right-hand side with respect to t using the product rule, with a M1

factor $\frac{dN}{dt}$ on each N term

$$\frac{d^2N}{dt^2} = 0.02 \left[(60 - N) \frac{dN}{dt} - N \frac{dN}{dt} \right] \quad \text{A1}$$

$$= 0.02(60 - 2N) \frac{dN}{dt} \quad \text{A1}$$

AG

METHOD 2: expands to $1.2N - 0.02N^2$ first, then $\frac{d^2N}{dt^2} = (1.2 - 0.04N) \frac{dN}{dt}$ (M1A1), then factorises to the given form (A1).

(b)

$$\text{sets } \frac{d^2N}{dt^2} = 0 \quad \text{M1}$$

$$N = 30 \text{ (since } \frac{dN}{dt} \neq 0 \text{ for } 0 < N < 60\text{)} \quad \text{A1}$$

$\frac{dN}{dt} > 0$ on the domain, so $\frac{d^2N}{dt^2} > 0$ for $N < 30$ and $\frac{d^2N}{dt^2} < 0$ for $N > 30$; the rate R1
increases then decreases, hence a maximum

Do not award R0A1. Accept an argument that $\frac{dN}{dt} = 0.02N(60 - N)$ is a quadratic in N with negative leading coefficient and vertex at $N = 30$.

(c)

$$\frac{dN}{dt} = 0.02(30)(30) = 18 \text{ (thousand per hour)}$$

A1

Units not required for the mark.

(d)

$$\text{separates variables: } \int \frac{1}{N(60-N)} dN = \int 0.02 dt$$

M1

$$\frac{1}{N(60-N)} = \frac{1}{60} \left(\frac{1}{N} + \frac{1}{60-N} \right)$$

A1

$$\frac{1}{60} \ln \left(\frac{N}{60-N} \right) = 0.02t + c \text{ (or } \ln \left(\frac{N}{60-N} \right) = 1.2t + k)$$

A1

$$\text{uses } N = 10 \text{ when } t = 0: k = \ln \frac{1}{5}, \text{ so } \frac{N}{60-N} = \frac{1}{5} e^{1.2t}$$

M1

$$5N = (60 - N)e^{1.2t} \Rightarrow N = \frac{60e^{1.2t}}{5 + e^{1.2t}} = \frac{60}{1 + 5e^{-1.2t}}$$

A1

AG

Condone missing modulus signs since $0 < N < 60$. The final rearrangement must be shown for the last A1.

(e)

$$\text{sets } N = 30: 1 + 5e^{-1.2t} = 2$$

M1

$$t = \frac{\ln 5}{1.2} = 1.34 \text{ (hours)}$$

A1

FT from part (b) only if a value of N in the domain was obtained.