

Gravitational fields and orbits: study notes

IB Physics HL · D.1

This topic describes how masses attract each other through an inverse-square force, how that force is expressed as a field and a potential, and how it keeps satellites and planets in circular orbits. In the exam you are asked to derive orbital speed and Kepler's third law from force balance, to calculate field strength, potential, escape speed and orbital energies, to locate the point where two fields cancel, and to explain in words what happens to a satellite that experiences atmospheric drag. Multiple-choice items typically test ratios that can be done by inspection; structured questions chain a short derivation into numerical estimates with the data booklet constants.

1. Kepler's three laws (learn them, not in the booklet)

First law: each planet moves in an ellipse with the Sun at one focus. Second law: the line from the Sun to the planet sweeps out equal areas in equal times, so a planet moves fastest when closest to the Sun. Third law: the square of the orbital period is proportional to the cube of the mean orbital radius, $T^2 \propto r^3$, with the same constant for every body orbiting the same central mass. The syllabus restricts calculations to circular orbits, so in practice you use the third law as $T^2 = \frac{4\pi^2 r^3}{GM}$ after deriving it (see section 5). Multiple-choice distractors often swap the laws, for example describing equal areas as a statement about periods, so know each law in its own words.

2. Newton's law of gravitation and field strength

Two point masses attract each other with a force $F = \frac{Gm_1m_2}{r^2}$ along the line joining them, where r is the centre-to-centre distance. A uniform sphere behaves as if all its mass were at its centre, which is why planets and stars can be treated as point masses for any point outside their surface. Gravitational field strength is the force per unit mass on a small test mass: $g = \frac{F}{m} = \frac{GM}{r^2}$, a vector pointing towards the centre of M , in N kg^{-1} (equivalent to m s^{-2}). Field lines are drawn radially inwards; where they are closer together the field is stronger, and they never cross. For two bodies the resultant field at a point on the line joining them is the vector sum of the two contributions; the field is zero where $\frac{GM_1}{x^2} = \frac{GM_2}{(d-x)^2}$, which gives $\frac{d-x}{x} = \sqrt{\frac{M_2}{M_1}}$. Both force and field formulas are in the booklet; the null-point method must be learned.

3. Gravitational potential energy and potential (HL)

The gravitational potential energy of two point masses is $E_p = -\frac{Gm_1m_2}{r}$. It is defined as the work done by an external agent to assemble the system by bringing the masses from infinity to separation r ; because the force is attractive, the agent does negative work, so E_p is negative and the zero is at infinity. Gravitational potential is the potential energy per unit mass at a point: $V_g = -\frac{GM}{r}$, a scalar in J kg^{-1} , also zero at infinity and negative everywhere else. Potentials from several bodies simply add (no vector components), which is the standard route to the escape speed from a planet-moon pair. The formulas are in the booklet; the choice of zero at infinity and the sign reasoning are not, and 'explain why the potential is negative' is a common one-mark part.

4. Potential gradient, work and equipotential surfaces

Field strength is the negative gradient of potential: $g = -\frac{\Delta V_g}{\Delta r}$. On a graph of V_g against r the curve rises from large negative values towards zero as r increases, and the magnitude of its slope at any point equals g there. The work done by an external agent moving a mass m between two points is $W = m\Delta V_g$, independent of the path. Equipotential surfaces are surfaces of constant V_g ; around a spherical mass they are concentric spheres, always perpendicular to the field lines. Equal potential steps are drawn closer together near the mass because the field is stronger there, and no work is done moving along an equipotential. In exam questions you may be given the spacing of equipotentials and asked to estimate g as the potential difference divided by the radial spacing.

5. Circular orbits: orbital speed, Kepler's third law and total energy

For a satellite of mass m in a circular orbit of radius r , gravity provides the centripetal force: $\frac{GMm}{r^2} = \frac{mv^2}{r}$, so $v_{\text{orbital}} = \sqrt{\frac{GM}{r}}$ (booklet). Substituting $v = \frac{2\pi r}{T}$ gives $T^2 = \frac{4\pi^2}{GM}r^3$, the quantitative form of Kepler's third law (learn the derivation; it is a standard 'show that'). Kinetic energy is $E_k = \frac{1}{2}mv^2 = \frac{GMm}{2r}$ and potential energy is $E_p = -\frac{GMm}{r}$, so the total energy is $E = -\frac{GMm}{2r}$, which is not in the booklet. A higher orbit has larger (less negative) total energy, smaller kinetic energy and larger potential energy. The energy that must be supplied to move a satellite between orbits is the difference in total energies; the energy to place a satellite into orbit from rest on the surface of a non-rotating planet is $E_{\text{orbit}} - E_{\text{surface}} = -\frac{GMm}{2r} + \frac{GMm}{R}$.

6. Escape speed and the energy to escape

To escape, an object needs just enough kinetic energy to reach infinity with zero speed: $\frac{1}{2}mv^2 - \frac{GMm}{r} = 0$, giving $v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$ (booklet). The energy required is $\frac{GMm}{r}$, or $m|V_g|$ at the launch point. Note that $v_{\text{esc}} = \sqrt{2}v_{\text{orbital}}$ at the same radius; the factor of two comes from the potential energy being twice the orbital kinetic energy in magnitude. For an object already in orbit the extra energy needed to escape is $\frac{GMm}{2r}$. When two bodies contribute, use the total potential: $v_{\text{esc}} = \sqrt{2|V_{\text{total}}|}$.

7. Qualitative effect of atmospheric drag on an orbit

A satellite in a low orbit meets a thin atmosphere. Drag does negative work on the satellite, so its total energy $E = -\frac{GMm}{2r}$ becomes more negative, which requires r to decrease: the orbit decays to a lower height. Since $v = \sqrt{\frac{GM}{r}}$, the speed increases as the radius falls. The kinetic energy rises, but the potential energy falls by twice as much, so the total still decreases. In an 'explain' part the marks are typically for: drag removes energy; total energy falls so radius falls; smaller radius means greater orbital speed. Stating that the satellite slows down is the classic wrong answer.

Formulas

$F = \frac{Gm_1m_2}{r^2}$	in the booklet
$g = \frac{F}{m} = \frac{GM}{r^2}$	in the booklet
$E_p = -\frac{Gm_1m_2}{r}$	in the booklet (HL only)
$V_g = -\frac{GM}{r}$	in the booklet (HL only)
$g = -\frac{\Delta V_g}{\Delta r}$	in the booklet (HL only)
$W = m\Delta V_g$	in the booklet (HL only)
$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$	in the booklet (HL only)
$v_{\text{orbital}} = \sqrt{\frac{GM}{r}}$	in the booklet (HL only)
$T^2 = \frac{4\pi^2}{GM}r^3$	learn it
$E_{\text{total}} = -\frac{GMm}{2r}$	learn it
$\frac{d-x}{x} = \sqrt{\frac{M_2}{M_1}}$ (null point)	learn it
$V_g = 0$ at infinity; the magnitude of g equals the gradient of the V_g against r graph	learn it

Worked examples

Example 1. A moon orbits a planet in a circular orbit of radius 4.2×10^8 m with a period of 1.77 days. (a) Show that the orbital speed of a satellite in a circular orbit of radius r around a mass M is $v = \sqrt{GM/r}$. (b) Show that $T^2 = \frac{4\pi^2 r^3}{GM}$. (c) Determine the mass of the planet.

1. (a) Gravity is the only force and acts as the centripetal force: $\frac{GMm}{r^2} = \frac{mv^2}{r}$.

2. Cancel m and one factor of r : $v^2 = \frac{GM}{r}$, so $v = \sqrt{\frac{GM}{r}}$.

3. (b) For a circular orbit $v = \frac{2\pi r}{T}$. Substitute: $\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$.

4. Rearrange: $T^2 = \frac{4\pi^2 r^3}{GM}$, which is Kepler's third law with constant $\frac{4\pi^2}{GM}$.

5. (c) Convert the period: $T = 1.77 \times 24 \times 3600 = 1.53 \times 10^5$ s.

6. $M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 (4.2 \times 10^8)^3}{6.67 \times 10^{-11} \times (1.53 \times 10^5)^2} = \frac{2.92 \times 10^{27}}{1.56} = 1.87 \times 10^{27}$ kg.

Answer: $M \approx 1.9 \times 10^{27}$ kg.

Example 2. A 1200 kg satellite is moved from a circular orbit of radius 7.0×10^6 m to a circular orbit of radius 4.2×10^7 m around the Earth ($M = 5.97 \times 10^{24}$ kg). (a) Calculate the orbital speed in the lower orbit. (b) Calculate the energy that must be supplied to change orbit. (c) State how the kinetic energy and the potential energy of the satellite change.

- Useful product: $GM = 6.67 \times 10^{-11} \times 5.97 \times 10^{24} = 3.98 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$.
- (a) $v = \sqrt{\frac{GM}{r_1}} = \sqrt{\frac{3.98 \times 10^{14}}{7.0 \times 10^6}} = \sqrt{5.69 \times 10^7} = 7.5 \times 10^3 \text{ m s}^{-1}$.
- (b) Total energy in orbit is $E = -\frac{GMm}{2r}$, so $\Delta E = \frac{GMm}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$.
- $GMm = 3.98 \times 10^{14} \times 1200 = 4.78 \times 10^{17} \text{ J m}$.
- $\frac{1}{r_1} - \frac{1}{r_2} = 1.43 \times 10^{-7} - 2.38 \times 10^{-8} = 1.19 \times 10^{-7} \text{ m}^{-1}$.
- $\Delta E = \frac{4.78 \times 10^{17} \times 1.19 \times 10^{-7}}{2} = 2.8 \times 10^{10} \text{ J}$.
- (c) Speed in the higher orbit is $\sqrt{\frac{3.98 \times 10^{14}}{4.2 \times 10^7}} = 3.1 \times 10^3 \text{ m s}^{-1}$, so kinetic energy decreases; potential energy becomes less negative, increasing by $GMm \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = 5.7 \times 10^{10} \text{ J}$, twice the total change.

Answer: (a) $7.5 \times 10^3 \text{ m s}^{-1}$; (b) $2.8 \times 10^{10} \text{ J}$ must be supplied; (c) kinetic energy falls, potential energy rises by twice the energy supplied.

Example 3. The Earth ($M_E = 5.97 \times 10^{24} \text{ kg}$, $R_E = 6.37 \times 10^6 \text{ m}$) and the Moon ($M_M = 7.35 \times 10^{22} \text{ kg}$) are separated by $d = 3.84 \times 10^8 \text{ m}$. (a) Determine the distance from the centre of the Earth to the point P on the line joining the two bodies where the resultant gravitational field is zero. (b) Show that the gravitational potential at P is about $-1.3 \times 10^6 \text{ J kg}^{-1}$. (c) Estimate the minimum launch speed from the Earth's surface for a probe to just reach P.

- (a) At P the two field strengths are equal in magnitude: $\frac{GM_E}{x^2} = \frac{GM_M}{(d-x)^2}$.
- Take square roots: $\frac{d-x}{x} = \sqrt{\frac{M_M}{M_E}} = \sqrt{0.0123} = 0.111$.
- $x = \frac{d}{1.111} = \frac{3.84 \times 10^8}{1.111} = 3.46 \times 10^8 \text{ m}$ (so P is $3.8 \times 10^7 \text{ m}$ from the Moon).
- (b) Potentials add as scalars: $V_P = -G \left(\frac{M_E}{x} + \frac{M_M}{d-x} \right) = -6.67 \times 10^{-11} \left(\frac{5.97 \times 10^{24}}{3.46 \times 10^8} + \frac{7.35 \times 10^{22}}{3.8 \times 10^7} \right)$.
- $= -6.67 \times 10^{-11} (1.73 \times 10^{16} + 1.9 \times 10^{15}) = -1.28 \times 10^6 \text{ J kg}^{-1}$, which is about $-1.3 \times 10^6 \text{ J kg}^{-1}$.
- (c) At the Earth's surface, $V_S = -6.67 \times 10^{-11} \left(\frac{5.97 \times 10^{24}}{6.37 \times 10^6} + \frac{7.35 \times 10^{22}}{3.78 \times 10^8} \right) = -6.25 \times 10^7 \text{ J kg}^{-1}$ (the Moon's term is negligible).
- Energy conservation per unit mass, arriving at P with zero speed: $\frac{1}{2}v^2 = V_P - V_S = 6.25 \times 10^7 - 1.3 \times 10^6 = 6.12 \times 10^7 \text{ J kg}^{-1}$.
- $v = \sqrt{2 \times 6.12 \times 10^7} = 1.1 \times 10^4 \text{ m s}^{-1}$, slightly less than the escape speed $\sqrt{2GM_E/R_E} = 1.12 \times 10^4 \text{ m s}^{-1}$, as expected.

Answer: (a) $3.46 \times 10^8 \text{ m}$ from the Earth's centre; (b) $V_P \approx -1.3 \times 10^6 \text{ J kg}^{-1}$; (c) about $1.1 \times 10^4 \text{ m s}^{-1}$.

Common errors

- Using altitude instead of orbital radius. Every r in D.1 is measured from the centre of the attracting body, so add the planet's radius to a given height before substituting.
- Dropping the minus sign on potential or potential energy, or stating that potential increases as you approach a mass. Potential is negative everywhere and rises towards zero at infinity; the field strength is the magnitude of its gradient.
- Squaring a ratio when the question asks for a difference, or vice versa. Ratios of g at two radii follow the inverse square; differences of V_g follow $1/r$ and must be calculated term by term.
- Choosing the wrong distance at the null point. The field from each body uses its own distance to the point (x and $d - x$); do not use d for both, and remember potential does not vanish there.
- Forgetting to convert periods into seconds and kilometres into metres before using $T^2 = 4\pi^2 r^3 / GM$, which produces a central mass wrong by many orders of magnitude.
- Claiming that drag slows a satellite. Drag reduces total energy, the radius falls, and the orbital speed $\sqrt{GM/r}$ increases; kinetic energy rises while potential energy falls by twice as much.

Before the exam, can you...

- Before the exam, can you state Kepler's three laws in your own words and derive $T^2 = 4\pi^2 r^3 / GM$ from $GMm/r^2 = mv^2/r$ in three lines?
- Can you define field strength as force per unit test mass, draw radial field lines and concentric equipotentials, and explain why their spacing changes with r ?
- Can you explain why $E_p = -GMm/r$ is negative, with the zero at infinity, and use $W = m\Delta V_g$ and $g = -\Delta V_g / \Delta r$ from a graph or from equipotential spacing?
- Can you find the total energy $-GMm/2r$ of a circular orbit, the energy to change orbits, the energy to launch into orbit from a non-rotating surface, and the energy to escape?
- Can you locate the null point between two masses, add their potentials to get an escape speed, and explain in words what atmospheric drag does to orbital radius and speed?