Galilean and special relativity: study notes

IB Physics HL · A.5 (HL)

This topic compares the Galilean rules that link two inertial frames with the Lorentz rules that replace them when speeds approach c. The exam asks you to compute the Lorentz factor, decide which observer measures the proper time or proper length, apply velocity addition, use the Lorentz transformation of a time or position, and draw or read a spacetime diagram to settle the order of two events. Marks are lost more often on attaching gamma to the wrong observer than on arithmetic.

1. Reference frames and Galilean transformations

A reference frame is a set of coordinate axes together with synchronised clocks, used to assign a position and a time to every event. An inertial frame is one in which Newton's first law holds: a body with no net force stays at rest or moves with constant velocity. Any frame moving at constant velocity relative to an inertial frame is also inertial. Take frame S′ moving at speed v along the x axis of frame S, with origins coinciding at $t = 0$. Newton assumed absolute time, so the Galilean transformation is $x' = x - vt$, $t' = t$ and, by differentiation, $u' = u - v$ (all in the booklet). These rules fail for light: a flash measured at c in S should appear at $c - v$ in S′, yet every experiment measures c in both frames. Because the velocity rule follows directly from absolute time, the whole transformation must be replaced.

2. The two postulates and the Lorentz transformations

Learn the postulates in words (not in the booklet): (1) the laws of physics are the same in every inertial frame; (2) the speed of light in vacuum is c for all inertial observers, independent of the motion of source or observer. Consequence: c is a limiting speed, since $\gamma = 1/\sqrt{1 - v^2/c^2}$ grows without bound as $v \to c$ and is not a real number for $v > c$. The Lorentz transformations in the booklet are $x' = \gamma(x - vt)$ and $t' = \gamma(t - vx/c^2)$; they also apply to intervals, $\Delta x' = \gamma(\Delta x - v\Delta t)$ and $\Delta t' = \gamma(\Delta t - v\Delta x/c^2)$. The inverse (learn it) swaps the sign of v and exchanges primed and unprimed symbols: $x = \gamma(x' + vt')$, $t = \gamma(t' + vx'/c^2)$. For $v \ll c$, $\gamma \to 1$ and the Galilean rules return. Memorise the handy pairs $0.6c \to \gamma = 1.25$, $0.8c \to \gamma = 5/3$ (both exact) and $0.995c \to \gamma \approx 10$ (more precisely 10.01).

3. Relativistic velocity addition

If an object moves at u in S and frame S′ moves at v relative to S (both along x), then in S′ the object moves at $u' = (u - v)/(1 - uv/c^2)$ (booklet). Rearranged, $u = (u' + v)/(1 + u'v/c^2)$: to combine two velocities, add them and divide by $1 + u'v/c^2$. Setting $u = c$ gives $u' = c$ whatever v is, which is the second postulate built into the formula. The result never exceeds c. Keep signs: velocities in the $-x$ direction are negative. Show the substitution clearly and express the final answer as a fraction of c.

4. Proper time, time dilation, proper length, length contraction

Definitions to learn. Proper time Δt_0: the time between two events measured in the frame in which the events happen at the same place, so a single clock is present at both. Every other inertial observer measures a longer interval, $\Delta t = \gamma \Delta t_0$ (booklet). Proper length L_0: the length of an object measured in the frame in which it is at rest; its ends can be read at leisure. An observer for whom the object moves must locate both ends at the same instant and measures $L = L_0/\gamma$ (booklet). To decide who holds the proper quantity, ask two questions: who has one clock at both events, and who sees the object at rest.

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A spacecraft crew measures the proper time of their own journey but measures a contracted Earth-to-star distance; Earth measures the proper distance but a dilated journey time. Always state explicitly which quantity is the proper one and which observer measures it; this is the reasoning examiners look for.

5. The invariant interval and the relativity of simultaneity

For two events, $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$ (booklet) has the same value in every inertial frame, even though Δt and Δx change. If $(\Delta s)^2 > 0$ (timelike) a frame exists in which the events are at the same place, and $\Delta s/c$ is the proper time between them; one event can cause the other. If $(\Delta s)^2 < 0$ (spacelike) a frame exists in which they are simultaneous and no frame has them at the same place; their time order depends on the observer and neither can cause the other. If $(\Delta s)^2 = 0$ they are connected by light. Simultaneity: events with $\Delta t = 0$ in S but $\Delta x \neq 0$ have $\Delta t' = -\gamma v \Delta x/c^2 \neq 0$ in S′. For an observer moving in the $+x$ direction, the event at larger x occurs first. Standard illustration: a flash at the centre of a moving carriage reaches both ends together for the passengers, but the platform observer sees the rear wall lit first because it moves towards the light.

6. Spacetime diagrams

Plot ct vertically and x horizontally for frame S, using the same scale so light travels along lines at 45°. A worldline is the path of an object through spacetime: vertical for an object at rest, a straight line steeper than 45° for constant velocity $v < c$ (gradient c/v), curved if accelerating. The light cone through an event consists of the two 45° lines; only events inside or on the future cone can be influenced by it, only events inside or on the past cone can have caused it. The axes of frame S′ moving at $+v$ (learn it, with $\tan\theta = v/c$ in the booklet): the ct' axis is the worldline of the S′ origin, rotated from the ct axis towards the $+x$ axis by θ; the x' axis is rotated from the x axis towards the ct axis by the same θ, so the pair closes symmetrically on the light line. If v is negative the axes lean the other way. Lines parallel to the x' axis are lines of simultaneity in S′; lines parallel to ct' are lines of constant x'. Scales on the tilted axes are stretched, so read values by projecting parallel to the other axis, not by measuring with a ruler. Simultaneity: draw a line parallel to x' through one event; the second event is earlier in S′ if it lies below that line, later if above. Time dilation: one tick of a moving clock sits on ct', and projecting it parallel to x onto the ct axis gives a longer time. Length contraction: a rod at rest in S′ is a band between two worldlines parallel to ct'; its length in S is the width of the band cut along the x axis, shorter than the proper length cut along x'.

7. Muon decay as evidence

Muons are created by cosmic rays several kilometres up in the atmosphere, moving at about $0.99c$ or faster, with a proper mean lifetime of about 2.2 μs. Without relativity they would travel only about 660 m in one lifetime, so almost none would reach sea level. Detectors on the ground record far more than this prediction. Earth-frame explanation: the muon clock runs slow, the lifetime is dilated by γ, so more muons survive the trip. Muon-frame explanation: the atmosphere moving past is contracted by γ, so the distance to the ground is short. Both frames agree on the number reaching the ground, which is what the experiment counts. Comparing counts at altitude and at sea level matches the relativistic prediction, not the classical one. A two-mark explanation typically needs the cause (dilation or contraction by γ) and the consequence (more survive).

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Formulas

$x' = x - vt, \quad t' = t, \quad u' = u - v$	in the booklet
$\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$	in the booklet
$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \dfrac{vx}{c^2}\right)$	in the booklet
$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \dfrac{vx'}{c^2}\right)$ (inverse)	learn it
$u' = \dfrac{u - v}{1 - uv/c^2}$	in the booklet
$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$	in the booklet
$\Delta t = \gamma \Delta t_0, \quad L = L_0/\gamma$	in the booklet
$\tan\theta = v/c$ (angle of the moving frame's axes)	in the booklet
The two postulates; definitions of proper time and proper length; which way the axes lean	learn it

Worked examples

Example 1. A spacecraft travels at $0.80c$ relative to Earth towards a star that is 12 light years away in the Earth frame. (a) Calculate γ. (b) Determine the journey time in the Earth frame and in the spacecraft frame, and identify which is the proper time. (c) Calculate the Earth-star distance measured by the crew. (d) A probe is launched forwards from the spacecraft at $0.50c$ relative to the spacecraft. Determine the probe speed in the Earth frame.

1. (a) $\gamma = 1/\sqrt{1 - 0.80^2} = 1/\sqrt{0.36} = 1/0.60 = 1.67$ (exactly $5/3$).
2. (b) Earth frame: $\Delta t = 12\text{ ly}/0.80c = 15$ years. The crew's clock is present at both departure and arrival, so the crew measure the proper time: $\Delta t_0 = \Delta t/\gamma = 15/(5/3) = 9.0$ years.
3. (c) 12 ly is the proper length (Earth and star at rest in the Earth frame). Crew measure $L = L_0/\gamma = 12 \times 0.60 = 7.2$ ly. Check: $7.2/0.80 = 9.0$ years, consistent with (b).
4. (d) Combine velocities: $u = (0.50c + 0.80c)/(1 + 0.50 \times 0.80) = 1.30c/1.40 = 0.93c$ (Galilean addition would wrongly give $1.3c$).

Answer: $\gamma = 1.67$; Earth time 15 years, spacecraft (proper) time 9.0 years; contracted distance 7.2 ly; probe speed $0.93c$.

Example 2. Muons are produced at a height of 3.0 km above a detector and travel vertically downwards at $0.995c$. The half-life of muons at rest is 1.56 μs. (a) Show that the time taken to reach the detector in the Earth frame is about 10 μs. (b) Calculate γ and hence the time taken in the muon frame. (c) Estimate the fraction of muons reaching the detector according to relativity, and compare with the classical prediction. (d) Outline how a muon-frame observer explains the result.

1. (a) $\Delta t = 3000/(0.995 \times 3.00 \times 10^8) = 1.005 \times 10^{-5}$ s ≈ 10 μs.
2. (b) $\gamma = 1/\sqrt{1 - 0.995^2} = 1/\sqrt{0.00998} \approx 10$. The muon carries its own clock, so its time is the proper time: $\Delta t_0 = 10.05/10 = 1.0$ μs.
3. (c) Relativistic: $1.0/1.56 = 0.64$ half-lives, fraction $= 2^{-0.64} \approx 0.64$. Classical: $10.05/1.56 = 6.4$ half-lives, fraction $= 2^{-6.4} \approx 0.012$. Relativity predicts roughly fifty times more muons, as observed.

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4. (d) In the muon frame the muon is at rest and the atmosphere rushes past at $0.995c$; the 3.0 km is contracted to $L = 3000/10 = 300$ m, covered in $300/(0.995c) = 1.0$ μs, the same proper time. Both frames agree on the surviving fraction.

Answer: $\gamma = 10$; Earth-frame time 10 μs, muon (proper) time 1.0 μs; about 64% survive relativistically against about 1% classically; the muon sees a contracted 300 m atmosphere.

Example 3. In frame S, event A occurs at $x = 0$, $t = 0$ and event B at $x = 600$ m, $t = 1.0$ μs. Frame S′ moves at $0.60c$ in the $+x$ direction, origins coinciding at $t = t' = 0$. (a) Calculate the time of B in S′ and state which event occurs first in S′. (b) Show that the spacetime interval between A and B is the same in both frames and comment on whether A could have caused B. (c) Describe how the order of the events is read from a spacetime diagram of S with the S′ axes drawn on it.

1. (a) $\gamma = 1/\sqrt{1 - 0.36} = 1.25$. $t'_B = 1.25\,(1.0 \times 10^{-6} - 0.60 \times 600/(3.00 \times 10^8)) = 1.25\,(1.0 - 1.2) \times 10^{-6} = -0.25$ μs. Since $t'_A = 0$, B occurs before A in S′.

2. (b) $x'_B = 1.25\,(600 - 0.60 \times 3.00 \times 10^8 \times 1.0 \times 10^{-6}) = 1.25 \times 420 = 525$ m. In S: $(c\Delta t)^2 - (\Delta x)^2 = 300^2 - 600^2 = -2.7 \times 10^5$ m^2. In S′: $(-75)^2 - 525^2 = 5625 - 275625 = -2.7 \times 10^5$ m^2. Equal, as the interval is invariant. It is negative (spacelike), so no signal could travel from A to B: A cannot have caused B, which is why their order can differ between frames.

3. (c) Draw ct vertical and x horizontal. The x' axis leaves the origin at angle θ above the x axis with $\tan\theta = 0.60$, so $\theta = 31°$; the ct' axis leaves at 31° from the ct axis towards $+x$. The x' axis is the line of simultaneity of S′ through A, with equation $ct = 0.60\,x$. Event B is at $(x, ct) = (600\text{ m}, 300\text{ m})$; the line through A at $x = 600$ m has $ct = 360$ m, so B lies below the line, hence earlier in S′.

Answer: $t'_B = -0.25$ μs, so B precedes A in S′; the interval is -2.7×10^5 m^2 in both frames (spacelike, no causal link); on the diagram B lies below the S′ line of simultaneity through A.

Common errors

- Assigning the proper time to the observer who watches the journey rather than the one whose single clock is present at both events. Always ask who is at both events before choosing whether to multiply or divide by gamma.
- Applying gamma in the wrong direction, for instance dividing a proper length by gamma to get something longer. Remember: moving clocks run slow (times get longer for everyone else), moving rods get shorter.
- Using Galilean addition near c, giving answers above c. As rough guidance, once speeds are around $0.1c$ or more the Galilean error reaches the percent level and grows quickly, so use the relativistic formula; a result greater than c is always wrong.
- Using only time dilation when the events are at different places in both frames. Then you need the full Lorentz transformation $\Delta t' = \gamma(\Delta t - v\Delta x/c^2)$, including the $v\Delta x/c^2$ term and its sign.
- Drawing the moving frame's axes wrongly: the ct′ and x′ axes must both lean towards the light line by the same angle with $\tan\theta = v/c$: for positive v the ct' axis leans towards $+x$ and the x' axis towards $+ct$. Drawing them at 45° or leaning the wrong way loses the diagram marks.
- Dropping units in gamma: v must be used as a fraction of c inside $\sqrt{1 - v^2/c^2}$, and lengths in light years with times in years keep speeds as fractions of c without conversion.

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Before the exam, can you...

- State the two postulates in words and explain why the Galilean velocity rule contradicts the second one?
- Compute gamma for any speed and quote the values for $0.6c$, $0.8c$ and $0.995c$ without a calculator?
- Define proper time and proper length and identify the correct observer in a spacecraft or muon scenario?
- Apply the Lorentz transformation and velocity addition with correct signs, and check with the invariant interval?
- Draw the axes of a moving frame at $\tan\theta = v/c$, sketch a light cone, and read simultaneity, time dilation and length contraction from the diagram?

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