Exponents and logarithms: practice worksheet

IB Mathematics: analysis and approaches HL · SL 1.1, 1.5, 1.7 · 48 minutes · 40 marks

Question 1 (*no calculator*)

(a) Find $\frac{(6.4\times10^{7})(3\times10^{-4})}{8\times10^{2}}$. Give your answer in the form $a \times 10^{k}$, where $1 \leq a < 10$ and k is an integer. [3]

Question 2 (*no calculator*)

(a) Find the exact value of $81^{-\frac{3}{4}}$. [2]

(b) Solve $5^{2x-1} = \frac{1}{125}$. [2]

Question 3 (*no calculator*)

Let $p = \log_5 2$ and $q = \log_5 3$.

(a) Find an expression for $\log_5\left(\frac{72}{25}\right)$ in terms of p and q. [2]

(b) Find an expression for $\log_{12}\left(\frac{125}{8}\right)$ in terms of p and q. [4]

Question 4 (*no calculator*)

Consider real values of x such that $x > 3$.

(a) Show that $(x-3)(x+1) = (x-1)^2 - 4$. [2]

(b) Hence, or otherwise, solve $\log_2(x-3) + 2\log_4(x+1) = 5$. [5]

Question 5 (*calculator*)

The number of cells in a culture is modelled by $N(t) = Ab^t$, where t is the time in hours, $A > 0$ and $b > 1$. The model gives $N(2) = 720$ and $N(5) = 2430$.

(a) Find b. [2]

(b) Find A and the time at which the model predicts 5000 cells. Give the time to three significant figures. [3]

(c) Determine the smallest non-negative integer n such that the model predicts more than 10000 cells after n hours. [3]

Question 6 (*no calculator*)

Let k be a real constant. Consider the equation $\log_3 x + \log_x 3 = k$, where $x > 0$ and $x \neq 1$. This question links logarithmic equations with quadratic equations and their discriminants.

(a) Let $u = \log_3 x$. Show that $u^2 - ku + 1 = 0$. [2]

(b) Hence solve the original equation when $k = \frac{10}{3}$. [3]

(c) Determine, with justification, the number of distinct real solutions of the original equation for each possible value of k. [4]

(d) The original equation has two solutions r and s satisfying $1 < r < s$ and $s = r^2$. Find the exact value of k. [3]

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Markscheme

Question 1

(a)

Separate the numerical coefficients and powers: $\frac{6.4\times3}{8}\times10^{7-4-2}$.	**M1**
Obtain the coefficient 2.4.	**A1**
Obtain 2.4×10^{1} in the required form.	**A1**

Accept an equivalent method using intermediate scientific notation.

Question 2

(a)

Use $81^{-\frac{3}{4}}=(81^{\frac{1}{4}})^{-3}=3^{-3}$.	**M1**
Obtain $\frac{1}{27}$.	**A1**

Accept equivalent applications of the laws of rational exponents.

(b)

Write $\frac{1}{125}=5^{-3}$ and equate exponents: $2x-1=-3$.	**M1**
Obtain $x=-1$.	**A1**

Accept a correct logarithmic method.

Question 3

(a)

Use logarithm laws to obtain $3\log_5 2+2\log_5 3-2\log_5 5$.	**M1**
Obtain $3p+2q-2$.	**A1**

Accept equivalent forms.

(b)

Apply change of base: $\frac{\log_5(125/8)}{\log_5 12}$.	**M1**
Obtain $\log_5(125/8)=3-3p$.	**A1**
Obtain $\log_5 12=2p+q$.	**A1**
Obtain $\frac{3-3p}{2p+q}$.	**A1**

Accept equivalent forms. The final expression must contain only p, q and numerical constants.

Question 4

(a)

Expand $(x-3)(x+1)$ to obtain x^2-2x-3.	**M1**
Rewrite $x^2-2x-3=(x^2-2x+1)-4$.	**A1**
Hence $(x-3)(x+1)=(x-1)^2-4$.	**AG**

Alternatively, expand the right-hand side and establish equality. Restating the given identity earns no marks.

(b)

Use change of base to obtain $2\log_4(x+1)=\log_2(x+1)$.	**M1**
Combine logarithms to obtain $\log_2((x-3)(x+1))=5$.	**M1**
Obtain $(x-3)(x+1)=32$, equivalently $(x-1)^2=36$.	**A1**
Obtain the candidates $x=7$ and $x=-5$.	**A1**
Reject $x=-5$ because $x>3$ is required, leaving $x=7$.	**R1**

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Accept expansion and factorisation of $x^2 - 2x - 35 = 0$. A candidate must be rejected with reference to the stated domain or an undefined logarithm.

Question 5

(a)

Divide $Ab^5 = 2430$ by $Ab^2 = 720$ to obtain $b^3 = \frac{2430}{720} = \frac{27}{8}$.	**M1**
Obtain $b = \frac{3}{2}$.	**A1**

Accept 1.5 or 1.50. Retain sufficient accuracy in subsequent calculations.

(b)

Obtain $A = \frac{720}{(3/2)^2} = 320$.	**A1**
Use logarithms to obtain $t = \frac{\ln(5000/320)}{\ln(3/2)}$.	**M1**
Obtain $t = 6.78$ hours to three significant figures.	**A1**

Accept change of base using any valid logarithm base. Allow FT from an earlier value of b greater than 1, provided A and the time are calculated consistently.

(c)

Use $320(3/2)^n > 10000$ to obtain $n > \frac{\ln(10000/320)}{\ln(3/2)}$.	**M1**
Obtain the boundary value $\frac{\ln(125/4)}{\ln(3/2)} \approx 8.49$.	**A1**
Conclude that the smallest integer satisfying the strict inequality is $n = 9$.	**R1**

Alternatively, award M1 for a systematic search, A1 for $N(8) = 8201.25$ and $N(9) = 12301.875$, and R1 for the justified minimum. Allow FT from earlier valid model parameters.

Question 6

(a)

Apply change of base to obtain $\log_x 3 = \frac{1}{u}$.	**M1**
Obtain $u + \frac{1}{u} = k$ and multiply by $u \neq 0$ to give $u^2 + 1 = ku$.	**A1**
Hence $u^2 - ku + 1 = 0$.	**AG**

The restriction $u \neq 0$ follows from $x \neq 1$. Merely quoting the given quadratic earns no marks.

(b)

Solve $3u^2 - 10u + 3 = 0$, for example by factorising as $(3u - 1)(u - 3) = 0$.	**M1**
Obtain $u = \frac{1}{3}$ or $u = 3$.	**A1**
Use $x = 3^u$ to obtain $x = \sqrt[3]{3}$ or $x = 27$.	**A1**

Accept $3^{1/3}$ for the cube root. Both solutions satisfy the original domain.

(c)

Consider the discriminant $\Delta = k^2 - 4$ of $u^2 - ku + 1 = 0$.	**M1**
For $k < -2$ or $k > 2$, obtain two distinct real roots u and hence two distinct solutions $x = 3^u$; neither root u is zero because their product is 1.	**A1**
For $k = 2$ or $k = -2$, obtain one repeated root u and hence one solution: respectively $x = 3$ or $x = \frac{1}{3}$.	**A1**
For $-2 < k < 2$, the discriminant is negative, so there are no real solutions.	**A1**

The justification must connect the roots in u to valid solutions in x. Accept equivalent discriminant arguments.

(d)

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Let $v = \log_3 r > 0$. Then $\log_3 s = 2v$, and the product of the quadratic roots gives $v(2v) = 1$. **M1**

Obtain $v = \frac{1}{\sqrt{2}}$, using $v > 0$. **A1**

Use the sum of the quadratic roots to obtain $k = v + 2v = \frac{3}{\sqrt{2}}$. **A1**

Alternatively, use $v + 1/v = 2v + 1/(2v)$, followed by $v > 0$. Accept equivalent exact forms.

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