Exponents and logarithms: study notes

IB Mathematics: analysis and approaches HL · SL 1.1, 1.5, 1.7

This topic is the algebra of powers and their inverses: writing numbers as $a \times 10^k$, manipulating integer and rational exponents, and using logarithms to undo exponentials. In the exam it appears as a short Paper 1 question worth 5 to 7 marks (simplify to a single logarithm, solve a log equation with a domain check, or solve an equation that is a hidden quadratic in a^x) and as a tool inside sequences, finance and modelling questions, where you are asked to find the smallest n or a parameter of an exponential model. Most marks are A marks for correct intermediate equations, so write each step as a clean equation.

1. Numbers of the form $a \times 10^k$

A number is in standard form when it is written as $a \times 10^k$ with $1 \le a < 10$ and k an integer. Multiply or divide such numbers by dealing with the two parts separately: $(3\times10^5)(4\times10^{-2}) = 12\times10^3 = 1.2\times10^4$, re-adjusting so that a stays in $[1, 10)$. To add or subtract, first write both numbers with the same power of ten: $5.1\times10^6 + 3\times10^5 = 5.1\times10^6 + 0.3\times10^6 = 5.4\times10^6$. A calculator will display results in this form; copy the exponent correctly and round a to the accuracy asked for (three significant figures unless told otherwise). None of this is in the booklet; it is expected as fluent arithmetic.

2. Laws of exponents (learn these, not in the booklet)

For a base $a > 0$ and any rational exponents m, n: $a^m a^n = a^{m+n}$, $\dfrac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$, $a^0 = 1$, $a^{-n} = \dfrac{1}{a^n}$, $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$. For example $8^{-2/3} = \dfrac{1}{(\sqrt[3]{8})^2} = \dfrac{1}{4}$ and $27^{4/3} = 3^4 = 81$. The most useful exam skill is rewriting different bases as powers of one base: $4^x = 2^{2x}$, $9^{x+1} = 3^{2x+2}$, $\sqrt{3} = 3^{1/2}$, $\dfrac{1}{25} = 5^{-2}$. Once every term is a power of the same base, an equation such as $2^{3x-1} = 4^{x+2}$ becomes $3x - 1 = 2x + 4$, so $x = 5$, with no logarithms needed.

3. What a logarithm is; base 10 and base e

The booklet gives the defining relationship $a^x = b \iff x = \log_a b$ for $a > 0$, $b > 0$, $a \neq 1$. In words, $\log_a b$ is the exponent to which a must be raised to give b: $\log_2 32 = 5$ because $2^5 = 32$, and $\log_5 \dfrac{1}{25} = -2$. Two consequences you must learn (not in the booklet): $\log_a 1 = 0$ and $\log_a a = 1$; the general identities $\log_a a^k = k$ and $a^{\log_a b} = b$ are in the booklet, but under Topic 2 (SL 2.9), so know them anyway. Base 10 is written $\log x$ and base e is written $\ln x$; both are calculator keys and are used for numerical evaluation, for example $\ln 7 \approx 1.95$ and $\log 250 \approx 2.40$. The domain condition is essential and the booklet only hints at it through the condition $b > 0$ beside the definition: $\log_a x$ exists only for $x > 0$. Questions state domains such as $x > 2$ so that laws may be applied, and markschemes usually award a mark for rejecting a root that makes any argument zero or negative. The graph of $y = \log_a x$ for $a > 1$ passes through $(1, 0)$, increases slowly, and has the y-axis as a vertical asymptote; it is the reflection of $y = a^x$ in the line $y = x$.

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4. Laws of logarithms and change of base (in the booklet)

For positive x, y and valid base a: $\log_a xy = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a x^m = m \log_a x$, and change of base $\log_b a = \frac{\log_c a}{\log_c b}$. Learn how to use them in both directions. Combining: $2\log_3 x - \log_3(x+1) = \log_3 \frac{x^2}{x+1}$. Splitting: $\ln \frac{\sqrt{x}}{e^2} = \frac{1}{2}\ln x - 2$. A constant can be written as a log in any base, $3 = \log_2 8$, which is often the key step when one side of an equation is a number. Change of base lets you evaluate $\log_3 20 = \frac{\ln 20}{\ln 3}$ on a calculator, and it gives the reciprocal result $\log_b a = \frac{1}{\log_a b}$ (take $c = a$). It is also how mixed-base equations are solved: $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2}\log_2 x$. A standard symbolic task gives $\log_a 2 = p$ and $\log_a 3 = q$ and asks for $\log_a 18 = p + 2q$ or $\log_a \frac{3}{\sqrt{2}} = q - \frac{1}{2}p$; break the argument into prime factors first. There is no law for $\log_a(x+y)$.

5. Solving exponential equations

If both sides can be written as powers of the same base, equate exponents. Otherwise take logarithms of both sides (any base, usually ln) and bring the exponent down with the power law: $5^x = 12 \Rightarrow x\ln 5 = \ln 12 \Rightarrow x = \frac{\ln 12}{\ln 5} \approx 1.54$, which is the same as $x = \log_5 12$. When the unknown appears in two different bases, collect the x terms: $3^{x+1} = 2^{2x}$ gives $(x+1)\ln 3 = 2x \ln 2$, so $x(2\ln 2 - \ln 3) = \ln 3$ and $x = \frac{\ln 3}{\ln 4 - \ln 3}$. In exact-answer questions leave the result as a quotient of logs or as a single log; a decimal with no working typically earns nothing. The same technique drives the 'smallest n such that' questions in sequences and finance: $2000(1.04)^n > 5000$ becomes $n \ln 1.04 > \ln 2.5$, so $n > 23.4$ and the smallest integer is $n = 24$. Remember that dividing by $\ln r$ with $0 < r < 1$ reverses the inequality because $\ln r < 0$. On calculator papers you may also solve numerically with a graph or table, but write the inequality down first for the method mark.

6. Hidden quadratics and log equations with a domain check

An equation such as $3^{2x} - 10 \cdot 3^x + 9 = 0$ is a quadratic in disguise: write $3^{2x} = (3^x)^2$, let $u = 3^x$, solve $u^2 - 10u + 9 = 0$ to get $u = 1$ or $u = 9$, then return to x: $3^x = 1 \Rightarrow x = 0$ and $3^x = 9 \Rightarrow x = 2$. A value $u \leq 0$ must be rejected because $a^x > 0$ for all real x. Watch for disguises: $4^x = (2^x)^2$, $e^{2x} = (e^x)^2$, $2^{x+1} = 2 \cdot 2^x$ and $9^x - 3^{x+1} = (3^x)^2 - 3 \cdot 3^x$. Similarly $(\log_2 x)^2 - 3\log_2 x + 2 = 0$ with $u = \log_2 x$ gives $u = 1$ or $u = 2$, so $x = 2$ or $x = 4$; note that $(\log x)^2$ is not $\log x^2$. For equations containing several logs, use the laws to reach a single log equal to a constant or to another single log, remove the logs with the definition, solve the resulting polynomial, and finally check every root against the original domain, writing a reason such as 'reject $x = -2$ since $\log_2 x$ is undefined'. That written reason is normally a separate mark.

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Formulas

$a^x = b \iff x = \log_a b$, with $a > 0, b > 0, a \neq 1$	in the booklet
$\log_a xy = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$,	in the booklet
$\log_a x^m = m \log_a x$	
$\log_b a = \frac{\log_c a}{\log_c b}$	in the booklet
$a^m a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$	learn it
$a^0 = 1$, $a^{-n} = \frac{1}{a^n}$, $a^{1/n} = \sqrt[n]{a}$, $a^{m/n} = (\sqrt[n]{a})^m$	learn it
$\log_a 1 = 0$, $\log_a a = 1$, $\log_a a^k = k$, $a^{\log_a b} = b$	learn the first two; others: booklet
$\log_a x$ is defined only for $x > 0$; $a^x > 0$ for all real x	learn it
$\log_b a = \frac{1}{\log_a b}$ (change of base with $c = a$)	learn it

Worked examples

Example 1. Solve $\log_2 x + \log_2(x-6) = 4$, where $x > 6$. (5 marks, non-calculator)

1. Combine the left side with the product law: $\log_2 (x(x-6)) = 4$. (M1 for the law applied)
2. Remove the logarithm using the definition $a^x = b \iff x = \log_a b$: $x(x-6) = 2^4 = 16$. (A1)
3. Expand and rearrange into a quadratic: $x^2 - 6x - 16 = 0$, which factorises as $(x-8)(x+2) = 0$. (A1 for the quadratic)
4. The candidate roots are $x = 8$ and $x = -2$.
5. Check the domain: $x = -2$ is rejected because $\log_2 x$ (and $\log_2(x-6)$) is undefined for a negative argument, and the question requires $x > 6$. (A1 for the rejection with a reason)
6. The only solution is $x = 8$; verify: $\log_2 8 + \log_2 2 = 3 + 1 = 4$. (A1)

Answer: $x = 8$

Example 2. Solve $2^{2x+1} - 9(2^x) + 4 = 0$, giving your answers exactly. (6 marks, non-calculator)

1. Rewrite the first term in terms of 2^x using the laws of exponents: $2^{2x+1} = 2 \cdot 2^{2x} = 2(2^x)^2$. (M1)
2. Let $u = 2^x$, so the equation becomes $2u^2 - 9u + 4 = 0$. (A1)
3. Factorise: $(2u-1)(u-4) = 0$, so $u = \frac{1}{2}$ or $u = 4$. (M1A1)
4. Both values are positive, so both are acceptable since $2^x > 0$ for all real x.
5. Return to x: $2^x = \frac{1}{2} = 2^{-1}$ gives $x = -1$. (A1)
6. $2^x = 4 = 2^2$ gives $x = 2$. (A1) Equivalently $x = \log_2 \frac{1}{2}$ and $x = \log_2 4$, which simplify to the same integers.

Answer: $x = -1$ or $x = 2$

Example 3. Let $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0, a \neq 1$. (a) Find $\log_a 12$ in terms of p and q. (b) Find $\log_a \frac{\sqrt{2}}{9}$ in terms of p and q. (c) Hence find $\log_6 a$ in terms of p and q. (7 marks, non-calculator)

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1. (a) Factorise the argument: $12 = 2^2 \times 3$, so $\log_a 12 = \log_a 2^2 + \log_a 3$. (M1)
2. Apply the power law: $\log_a 12 = 2\log_a 2 + \log_a 3 = 2p + q$. (A1)
3. (b) Write $\sqrt{2} = 2^{1/2}$ and $9 = 3^2$, then use the quotient law: $\log_a \dfrac{\sqrt{2}}{9} = \log_a 2^{1/2} - \log_a 3^2$. (M1)
4. Apply the power law to each term: $= \dfrac{1}{2}\log_a 2 - 2\log_a 3 = \dfrac{1}{2}p - 2q$. (A1)
5. (c) Use change of base with $c = a$: $\log_6 a = \dfrac{\log_a a}{\log_a 6} = \dfrac{1}{\log_a 6}$, since $\log_a a = 1$. (M1)
6. Now $\log_a 6 = \log_a 2 + \log_a 3 = p + q$. (A1)
7. Therefore $\log_6 a = \dfrac{1}{p+q}$. (A1)

Answer: (a) $2p + q$; (b) $\dfrac{1}{2}p - 2q$; (c) $\dfrac{1}{p+q}$

Common errors

- Inventing a law for the log of a sum, writing $\log_a(x + y) = \log_a x + \log_a y$. The laws only cover products, quotients and powers; if a sum appears inside a log, leave it as a single argument.
- Forgetting the domain check and giving two answers to a log equation. After removing the logs, substitute every root back into each original argument and state in words why a root is rejected; that sentence is usually a mark.
- Dropping the base when changing base, for example writing $\log_4 x = \log_2 x$ or $\dfrac{\ln 4}{\ln x}$. Keep the base as the denominator: $\log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{1}{2}\log_2 x$.
- Sign and fraction errors with exponents, such as $8^{-2/3} = -4$ or $a^{-n} = -a^n$. A negative exponent means a reciprocal, never a negative value; a fractional exponent means a root.
- Confusing $(\log x)^2$ with $\log x^2 = 2\log x$, or $\ln 2x$ with $2\ln x$. Use brackets carefully and substitute $u = \log x$ before solving a quadratic.
- Rounding too early in 'smallest n' or model-fitting questions. Keep $\dfrac{\ln 2.5}{\ln 1.04}$ exact or to several decimal places, then round only at the final line, and remember a smallest integer satisfying a strict inequality must be rounded up.

Before the exam, can you...

- Convert between $a^x = b$ and $x = \log_a b$ instantly, evaluate logs such as $\log_3 81$ and $\ln e^{-2}$ without a calculator, and state that $\log_a 1 = 0$ and $\log_a a = 1$?
- Write 4^x, 9^{x+1}, $\sqrt{3}$ and $\dfrac{1}{25}$ as single powers of 2, 3, 3 and 5, and use this to spot a hidden quadratic in a^x?

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- Combine $2\log_3 x - \log_3(x+1)$ into one log, and split $\ln\dfrac{\sqrt{x}}{e^2}$ into separate terms, quoting the booklet laws correctly?
- Solve $\log_2 x + \log_2(x-2) = 3$ completely, including rejecting the invalid root with a written reason?
- Take logs of both sides of $5^x = 12$ or $3^{x+1} = 2^{2x}$, solve for x exactly as a quotient of logs, and turn $A(1.04)^n > B$ into the smallest integer n?

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