Discrete random variables and the binomial distribution: practice worksheet

IB Mathematics: analysis and approaches HL · SL 4.7, 4.8, AHL 4.14 (discrete) · 53 minutes · 44 marks

Question 1 (*no calculator*)

The discrete random variable X has the following probability distribution:

x	-2	0	3
$P(X = x)$	$\frac{1}{4}$	k	$\frac{1}{2}$

(a) Find k. [2]

(b) Find $E(X)$. [2]

Question 2 (*no calculator*)

A fair spinner has two equal sectors, blue and gold. It is spun independently three times. Let X be the number of times it lands on blue.

(a) Write down the distribution of X, including its parameters. [1]

(b) Find $P(X = 3)$. [2]

(c) Write down $E(X)$. [1]

Question 3 (*no calculator*)

A festival game awards a prize of W credits. Its probability distribution is

w	0	2	8
$P(W = w)$	k	$2k$	$\frac{1}{4}$

The entry fee is 4 credits. A player's net gain is the prize minus the entry fee.

(a) Find k. [2]

(b) Find the expected prize. [2]

(c) Determine whether the game is fair at the stated entry fee, giving a reason. Hence find the entry fee that would make the game fair. [3]

Question 4 (*calculator*)

An automated sorter incorrectly labels a parcel with probability 0.25. Labelling outcomes are independent. Eight parcels are processed, and X is the number incorrectly labelled.

(a) State the distribution of X, including its parameters. [1]

(b) Find the probability that exactly three parcels are incorrectly labelled. [2]

(c) Find the probability that at least two parcels are incorrectly labelled. [2]

(d) Find the mean and the variance of X. [3]

Question 5 (*no calculator*)

A device records a discrete random variable X with distribution

x	-2	1	4
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Its displayed reading is $Y = 6 - 2X$.

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(a) Show that $E(X) = \frac{1}{2}$. [2]

(b) Find $\mathrm{Var}(X)$. [3]

(c) Hence find $E(Y)$ and $\mathrm{Var}(Y)$. [3]

(d) Without further calculation, explain why $Y + 7$ has the same variance as Y. [1]

Question 6 (*calculator*)

An electronic ticket costs 2 credits and independently wins a prize of 5 credits with probability 0.20. A losing ticket pays nothing. A player buys n tickets. Let X be the number of winning tickets and let G be the player's total net gain, so that $G = 5X - 2n$.

(a) State the distribution of X, including its parameters. [1]

(b) In the case where $n = 10$, find the probability that at least two tickets win. [2]

(c) Using logarithms, determine the least integer value of n for which the probability of at least one winning ticket is greater than 0.95. [4]

(d) The player buys the number of tickets found in part (c). Find $E(G)$ and $\mathrm{Var}(G)$. [3]

(e) The organiser replaces the prize of 5 credits with a prize of r credits. The ticket price and winning probability remain unchanged. Determine r so that the game is fair for any positive integer n. [2]

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Markscheme

Question 1

(a)

$\frac{1}{4} + k + \frac{1}{2} = 1.$ **M1**
$k = \frac{1}{4}.$ **A1**

Accept equivalent exact forms.

(b)

$E(X) = (-2)\left(\frac{1}{4}\right) + 0(k) + 3\left(\frac{1}{2}\right).$ **M1**
$E(X) = 1.$ **A1**

Question 2

(a)

$X \sim \mathrm{B}\left(3, \frac{1}{2}\right).$ **A1**

Both parameters and the binomial distribution must be identified.

(b)

$P(X = 3) = \left(\frac{1}{2}\right)^3.$ **M1**
$P(X = 3) = \frac{1}{8}.$ **A1**

Accept use of the binomial probability formula.

(c)

$E(X) = 3\left(\frac{1}{2}\right) = \frac{3}{2}.$ **A1**

Question 3

(a)

$k + 2k + \frac{1}{4} = 1.$ **M1**
$k = \frac{1}{4}.$ **A1**

(b)

$E(W) = 0(k) + 2(2k) + 8\left(\frac{1}{4}\right).$ **M1**
$E(W) = 3$ credits. **A1**

Award A1 FT for a correct evaluation using the candidate's value of k, provided the probabilities form a valid distribution.

(c)

$E(\text{net gain}) = E(W) - 4 = 3 - 4 = -1.$ **M1**
The game is not fair because the expected net gain is not zero. **R1**
The fair entry fee is 3 credits. **A1**

Accept that the player loses one credit per play on average. Allow FT from part (b) for a consistent conclusion and fair entry fee.

Question 4

(a)

$X \sim \mathrm{B}(8, 0.25).$ **A1**

(b)

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Use the binomial probability function with $n = 8$, $p = 0.25$ and $x = 3$, or evaluate $\binom{8}{3}(0.25)^3(0.75)^5$. **M1**
$P(X = 3) = 0.2076416016\ldots \approx 0.208$. **A1**

Accept the exact fraction.

(c)

$P(X \geq 2) = 1 - P(X \leq 1)$, using the binomial cumulative probability function. **M1**
$P(X \geq 2) = 0.6329193115\ldots \approx 0.633$. **A1**

Accept a direct sum of probabilities from 2 to 8, or a correctly specified calculator upper-tail calculation.

(d)

Use $E(X) = np$ and $\text{Var}(X) = np(1-p)$. **M1**
$E(X) = 8(0.25) = 2$. **A1**
$\text{Var}(X) = 8(0.25)(0.75) = 1.5$. **A1**

Award M1 for a correct substitution into either formula.

Question 5

(a)

$E(X) = (-2)\left(\frac{1}{3}\right) + 1\left(\frac{1}{2}\right) + 4\left(\frac{1}{6}\right)$. **M1**
$E(X) = \frac{-4+3+4}{6} = \frac{3}{6}$. **A1**
Hence $E(X) = \frac{1}{2}$. **AG**

The given answer alone earns no marks.

(b)

$E(X^2) = (-2)^2\left(\frac{1}{3}\right) + 1^2\left(\frac{1}{2}\right) + 4^2\left(\frac{1}{6}\right)$. **M1**
$E(X^2) = \frac{9}{2}$. **A1**
$\text{Var}(X) = \frac{9}{2} - \left(\frac{1}{2}\right)^2 = \frac{17}{4}$. **A1**

Accept direct evaluation of $\sum (x-\mu)^2 P(X = x)$ as an alternative method.

(c)

Use $E(Y) = 6 - 2E(X)$ and $\text{Var}(Y) = (-2)^2\,\text{Var}(X)$. **M1**
$E(Y) = 6 - 2\left(\frac{1}{2}\right) = 5$. **A1**
$\text{Var}(Y) = 4\left(\frac{17}{4}\right) = 17$. **A1**

Award M1 for either correct transformation rule. Allow A1 FT for the variance using the candidate's non-negative variance from part (b).

(d)

Adding a constant leaves variance unchanged, or $\text{Var}(Y + 7) = 1^2\,\text{Var}(Y)$. **R1**

A numerical answer without an explanation is insufficient.

Question 6

(a)

$X \sim \text{B}(n, 0.20)$. **A1**

(b)

$P(X \geq 2) = 1 - P(X \leq 1)$ for $X \sim \text{B}(10, 0.20)$. **M1**
$P(X \geq 2) = 0.6241903616 \approx 0.624$. **A1**

Accept a correctly specified direct upper-tail calculation.

(c)

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$1-(0.8)^n > 0.95$, giving $(0.8)^n < 0.05$. **M1**
Take logarithms to obtain $n\ln(0.8) < \ln(0.05)$. **M1**
$n > \frac{\ln(0.05)}{\ln(0.8)} = 13.4251\ldots$, reversing the inequality because $\ln(0.8) < 0$. **A1**
The least integer value is $n = 14$. **A1**

Accept any logarithm base. A calculator search alone does not earn the logarithm method mark.

(d)

Use $E(G) = 5(np) - 2n$ and $\text{Var}(G) = 25np(1-p)$ with $p = 0.20$. **M1**
$E(G) = 5(14)(0.20) - 28 = -14$ credits. **A1**
$\text{Var}(G) = 25(14)(0.20)(0.80) = 56$ credits squared. **A1**

Award M1 for either correct expression. Allow A1 FT for each result using the candidate's positive integer value from part (c).

(e)

For fairness, $E(rX - 2n) = 0$, giving $0.20nr - 2n = 0$. **M1**
Since $n > 0$, $r = 10$ credits. **A1**

Accept the equivalent per-ticket equation $0.20r = 2$.

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