Differentiation rules and standard derivatives: practice worksheet

IB Mathematics: analysis and approaches HL · SL 5.3, 5.6, 5.7, AHL 5.15 · 52 minutes · 43 marks

Question 1 (*no calculator*)

No calculator. Let $f(x) = 3x^{4/3} - 6x^{-1/2} + 5$, where $x > 0$.

(a) Find $f'(x)$. [2]

(b) Find $f''(1)$. [1]

Question 2 (*no calculator*)

No calculator. A curve has equation $y = e^x \sin(2x) + \cos x$.

(a) Find $\frac{dy}{dx}$. [3]

(b) Hence find the gradient of the curve at $x = 0$. [1]

Question 3 (*no calculator*)

No calculator. Let $f(x) = \ln\left(\frac{x+2}{2x+1}\right)$, where $x > 0$.

(a) Show that $f'(x) = -\frac{3}{(x+2)(2x+1)}$. [4]

(b) Hence find the gradient of the graph of f at $x = 1$. [1]

(c) Find $f''(1)$. [2]

Question 4 (*no calculator*)

No calculator. Let $u(x) = 2^{3x}$ and $v(x) = \log_2 x$, where $x > 0$. Define $g(x) = u(x)v(x)$.

(a) Write down $u'(x)$ and $v'(x)$. [2]

(b) Show that $g'(x) = 2^{3x}\left(3\ln x + \frac{1}{x\ln 2}\right)$. [3]

(c) Find $g''(1)$, giving your answer in exact form. [3]

Question 5 (*no calculator*)

No calculator. Let $h(x) = \frac{\sec x}{1+\arctan(\sin x)}$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Define $d(x) = 1 + \arctan(\sin x)$, so that $d(x)h(x) = \sec x$.

(a) Show that $h'(x) = \frac{\sec x\left[(1+\arctan(\sin x))\tan x - \frac{\cos x}{1+\sin^2 x}\right]}{(1+\arctan(\sin x))^2}$. [5]

(b) Hence find $h'(0)$. [1]

(c) By differentiating the identity $d(x)h(x) = \sec x$ twice, find $h''(0)$. [3]

Question 6 (*no calculator*)

No calculator. This question links differentiation to equations of perpendicular lines. For $x > 0$ and $k \in \mathbb{R}$, define $u(x) = \frac{x-1}{x+1}$ and $F_k(x) = \arcsin(u(x)) + k\arccos(u(x))$. The tangent to the graph of F_k at $x = 1$ is denoted by T_k.

(a) Show that $1 - [u(x)]^2 = \frac{4x}{(x+1)^2}$. [2]

(b) Hence show that $F_k'(x) = \frac{1-k}{(x+1)\sqrt{x}}$. [4]

(c) Find an equation of T_k in terms of k. [3]

(d) Given that T_k is perpendicular to the line $y = 2x - 3$, determine k and the x-intercept of T_k. [3]

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Markscheme

Question 1

(a)

Attempts to apply the power rule to both non-constant terms. **M1**

$f'(x) = 4x^{1/3} + 3x^{-3/2}$. **A1**

Accept equivalent forms.

(b)

$f''(1) = \frac{4}{3} - \frac{9}{2} = -\frac{19}{6}$. **A1**

FT their derivative from part (a), provided it contains a non-constant term and is correctly differentiated and evaluated.

Question 2

(a)

Attempts the product rule on $e^x \sin(2x)$. **M1**

$\frac{d}{dx}(e^x \sin(2x)) = e^x \sin(2x) + 2e^x \cos(2x)$. **A1**

$\frac{dy}{dx} = e^x \sin(2x) + 2e^x \cos(2x) - \sin x$. **A1**

Accept a correctly factorised equivalent expression.

(b)

The gradient is $1(0) + 2(1)(1) - 0 = 2$. **A1**

FT correct evaluation of their derivative from part (a).

Question 3

(a)

Writes $f(x) = \ln(x+2) - \ln(2x+1)$ and attempts to differentiate. **M1**

$f'(x) = \frac{1}{x+2} - \frac{2}{2x+1}$. **A1**

$f'(x) = \frac{(2x+1)-2(x+2)}{(x+2)(2x+1)}$. **A1**

Simplifies the numerator: $2x + 1 - 2x - 4 = -3$. **A1**

$f'(x) = -\frac{3}{(x+2)(2x+1)}$. **AG**

Alternative method: award M1 for attempting the chain and quotient rules, A1 for the inner derivative $-3/(2x+1)^2$, A1 for multiplying by $(2x+1)/(x+2)$, and A1 for cancelling factors. The simplification must be shown.

(b)

$f'(1) = -\frac{3}{3\times 3} = -\frac{1}{3}$. **A1**

(c)

Differentiates $-3(2x^2+5x+2)^{-1}$ to obtain $f''(x) = \frac{3(4x+5)}{(2x^2+5x+2)^2}$. **M1**

$f''(1) = \frac{27}{81} = \frac{1}{3}$. **A1**

For M1, accept a correct quotient-rule attempt with an error in simplification. Accept equivalent forms.

Question 4

(a)

$u'(x) = 3(\ln 2)2^{3x}$. **A1**

$v'(x) = \frac{1}{x \ln 2}$. **A1**

(b)

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Uses $g'(x) = u'(x)v(x) + u(x)v'(x)$. **M1**
$g'(x) = 3(\ln 2)2^{3x}\frac{\ln x}{\ln 2} + \frac{2^{3x}}{x\ln 2}$. **A1**
Cancels $\ln 2$ in the first term and factorises to obtain $2^{3x}\left(3\ln x + \frac{1}{x\ln 2}\right)$. **A1**
$g'(x) = 2^{3x}\left(3\ln x + \frac{1}{x\ln 2}\right)$. **AG**

The cancellation and factorisation must be evident.

(c)

Applies the product rule to the expression for $g'(x)$. **M1**
$g''(x) = 2^{3x}\left(9(\ln 2)\ln x + \frac{6}{x} - \frac{1}{x^2\ln 2}\right)$. **A1**
$g''(1) = 8\left(6 - \frac{1}{\ln 2}\right) = 48 - \frac{8}{\ln 2}$. **A1**

Accept an unsimplified correct expression for $g''(x)$. FT the final evaluation mark from their second derivative, provided the product rule has been attempted.

Question 5

(a)

Attempts the quotient rule with numerator $\sec x$ and denominator $d(x)$. **M1**
$\frac{d}{dx}(\sec x) = \sec x \tan x$. **A1**
$d'(x) = \frac{\cos x}{1+\sin^2 x}$. **A1**
$h'(x) = \frac{d(x)\sec x\tan x - \sec x\frac{\cos x}{1+\sin^2 x}}{d(x)^2}$. **A1**
Factorises $\sec x$ from the numerator and substitutes $d(x) = 1 + \arctan(\sin x)$. **A1**
$h'(x) = \frac{\sec x\left[(1+\arctan(\sin x))\tan x - \frac{\cos x}{1+\sin^2 x}\right]}{(1+\arctan(\sin x))^2}$. **AG**

Accept a product-rule approach using $h(x) = \sec x\,[d(x)]^{-1}$ with corresponding accuracy steps. The factorisation must be shown.

(b)

$h'(0) = \frac{1(1\times 0 - 1)}{1^2} = -1$. **A1**

(c)

Applies the product rule twice to obtain $dh'' + 2d'h' + d''h = \sec x\tan^2 x + \sec^3 x$. **M1**
Uses $d(0) = 1$, $d'(0) = 1$ and $d''(0) = 0$ to obtain $h''(0) + 2(-1) = 1$. **A1**
$h''(0) = 3$. **A1**

The value $d''(0) = 0$ follows by differentiating $d'(x) = \cos x/(1+\sin^2 x)$. FT their value from part (b) in the last two lines, giving $h''(0) = 1 - 2h'(0)$.

Question 6

(a)

Writes $1 - [u(x)]^2 = \frac{(x+1)^2 - (x-1)^2}{(x+1)^2}$. **M1**
Expands the numerator to obtain $x^2 + 2x + 1 - (x^2 - 2x + 1) = 4x$. **A1**
$1 - [u(x)]^2 = \frac{4x}{(x+1)^2}$. **AG**

Accept factorisation of the difference of two squares instead of expansion.

(b)

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Attempts the chain rule for both inverse trigonometric terms. **M1**
$u'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. **A1**
$F_k'(x) = \frac{u'(x)}{\sqrt{1-u(x)^2}} - \frac{ku'(x)}{\sqrt{1-u(x)^2}} = \frac{2(1-k)}{(x+1)^2\sqrt{1-u(x)^2}}$. **A1**
Since $x > 0$, uses $\sqrt{1-u(x)^2} = \frac{2\sqrt{x}}{x+1}$ and cancels factors to obtain $\frac{1-k}{(x+1)\sqrt{x}}$. **A1**
$F_k'(x) = \frac{1-k}{(x+1)\sqrt{x}}$. **AG**

The negative sign in the derivative of arccos must be evident. Accept first using $\arcsin u + \arccos u = \pi/2$, *with corresponding method and accuracy steps.*

(c)

Uses the tangent equation $y - F_k(1) = F_k'(1)(x-1)$. **M1**
Obtains $F_k(1) = \frac{k\pi}{2}$ and $F_k'(1) = \frac{1-k}{2}$. **A1**
$y - \frac{k\pi}{2} = \frac{1-k}{2}(x-1)$. **A1**

Accept equivalent forms of the line equation.

(d)

Uses perpendicular gradients to set $\frac{1-k}{2} = -\frac{1}{2}$. **M1**
$k = 2$. **A1**
Sets $y = 0$ in $y - \pi = -\frac{1}{2}(x-1)$ to obtain $x = 1 + 2\pi$. **A1**

FT the final mark from their value of k and their tangent equation, provided the resulting line has a unique x-intercept.

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