Differentiation rules and standard derivatives: study notes

IB Mathematics: analysis and approaches HL · SL 5.3, 5.6, 5.7, AHL 5.15

This topic is the toolkit of differentiation: the power rule, the standard derivatives of sin, cos, exp and ln (and at HL of the other trigonometric, exponential, logarithmic and inverse trigonometric functions), and the chain, product and quotient rules that combine them, together with second and higher derivatives. In the exam it is rarely a question on its own: it is the opening 2 to 4 marks of a longer question, often a 'show that' whose result is then used to find a gradient, a tangent, a stationary point or an integral. You earn the marks by naming the rule, differentiating every factor correctly and writing the simplification in full.

1. The power rule and polynomials (SL 5.3)

For integer n (extended to any rational n in SL 5.6), $\frac{d}{dx}x^n = nx^{n-1}$ (in the booklet). Combined with the facts that constants have derivative zero and that differentiation is linear ($\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$), this differentiates every polynomial term by term. The rule also covers negative and fractional powers, so before differentiating rewrite $\frac{1}{x^3}$ as x^{-3}, $\sqrt{x}$ as $x^{1/2}$ and $\frac{3}{\sqrt{x}}$ as $3x^{-1/2}$. For example $\frac{d}{dx}\left(4x^3 - \frac{2}{x} + \sqrt{x}\right) = 12x^2 + 2x^{-2} + \frac{1}{2}x^{-1/2}$. The derivative $f'(x)$ is the gradient function: $f'(a)$ is the gradient of the curve at $x = a$.

2. Standard derivatives: sin, cos, exp, ln (SL 5.6)

The four SL standard results are in the booklet, minus sign included: $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$. One thing is not in the booklet and must be known: x in $\sin x$ must be in radians for these results to hold. Even though the sign on the derivative of cosine is printed, it is easy to lose under pressure, so check it every time you write it down. The composite versions are also not in the booklet and are so common that they should be automatic: $\frac{d}{dx}e^{kx} = ke^{kx}$, $\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$, $\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$, $\frac{d}{dx}\ln(ax + b) = \frac{a}{ax+b}$. A useful preliminary step with logarithms is to apply the log laws first: $\ln\frac{x^2}{2x+1} = 2\ln x - \ln(2x + 1)$ (for $x > 0$) differentiates to $\frac{2}{x} - \frac{2}{2x+1}$ with no quotient rule needed.

3. The chain rule (SL 5.6)

If $y = g(u)$ and $u = f(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (in the booklet, in this form only), or in function notation $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$, which is not printed. Method: identify the outer function and the inner function, differentiate the outer leaving the inside unchanged, then multiply by the derivative of the inside. Examples: $\frac{d}{dx}(3x^2-1)^5 = 5(3x^2-1)^4 \cdot 6x = 30x(3x^2-1)^4$; $\frac{d}{dx}\sqrt{1-2x} = \frac{1}{2}(1-2x)^{-1/2} \cdot (-2) = -\frac{1}{\sqrt{1-2x}}$; $\frac{d}{dx}e^{\sin x} = \cos x\, e^{\sin x}$. Multi-layer chains are a favourite trap: $\frac{d}{dx}\ln(\cos 2x) = \frac{1}{\cos 2x} \cdot (-\sin 2x) \cdot 2 = -2\tan 2x$. Each layer contributes one factor; count the layers before you start.

4. Product rule, quotient rule and combinations (SL 5.6)

Both are in the booklet: $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. The order of subtraction in the quotient rule matters: denominator times derivative of numerator first. Write u, u', v, v' as a small table before substituting; the marks are typically M1 for attempting the correct rule, A1 for each correctly differentiated factor and A1 for the simplified form, so a clearly laid out table helps

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to secure the method mark even if the algebra later slips. Almost every exam derivative combines rules, for instance $y = x^2 \sin 3x$ needs the product rule with a chain rule inside ($v' = 3\cos 3x$), and $y = \frac{\ln x}{x^2}$ gives $\frac{x^2 \cdot \frac{1}{x} - 2x \ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$. For 'show that' questions, factorise out common factors such as e^{2x} or $(2x+1)$ and show the cancellation line by line; the markscheme ends with AG and expects to see the algebra, not just the answer.

5. Second and higher derivatives (SL 5.7)

Differentiating $f'(x)$ again gives the second derivative, written $f''(x)$ or $\frac{d^2y}{dx^2}$; further derivatives are $f'''(x)$, $f^{(n)}(x)$ or $\frac{d^n y}{dx^n}$. This notation is not in the booklet. The second derivative is the rate of change of the gradient and is used to classify stationary points and to discuss concavity ($f'' > 0$ concave up, $f'' < 0$ concave down), and higher derivatives appear at HL when building series or proving a pattern by induction. Practical advice: simplify $f'(x)$ fully before differentiating again, because an unsimplified product or quotient makes the second derivative far longer than necessary. For example $f(x) = xe^{-2x}$ gives $f'(x) = (1-2x)e^{-2x}$ and then $f''(x) = -2e^{-2x} - 2(1-2x)e^{-2x} = (4x-4)e^{-2x}$. Note that $f''(a) = 0$ alone does not prove a point of inflexion; a change of sign of f'' (or of concavity) is required.

6. AHL standard derivatives and their composites (AHL 5.15)

The booklet gives: $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec x = \sec x \tan x$, $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$, $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$, $\frac{d}{dx}a^x = a^x \ln a$, $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$, $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. Know by heart that the derivatives of the three 'co' functions carry a minus sign, and be able to derive $\tan x$ and $\sec x$ from the quotient rule in case a 'show that' asks for it: $\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$. Composites are not in the booklet and are examined often: $\frac{d}{dx}\arctan(3x) = \frac{3}{1+9x^2}$, $\frac{d}{dx}\arcsin(x^2) = \frac{2x}{\sqrt{1-x^4}}$, $\frac{d}{dx}\sec(2x) = 2\sec 2x \tan 2x$, $\frac{d}{dx}2^{x^2} = 2x \cdot 2^{x^2} \ln 2$. For a^x you can also rewrite as $e^{x \ln a}$ and use the chain rule.

7. Gradients at a point and exam presentation

Once $f'(x)$ is found, the gradient at $x = a$ is $f'(a)$: substitute, keep exact values on Paper 1 (leave e^2, $\frac{\pi}{4}$, $\sqrt{3}$ as they are) and on Paper 2 give the answer exact or correct to 3 significant figures unless the question states otherwise, keeping full accuracy in the working and rounding only at the end; rounding too early or giving too few figures loses marks. A tangent at $(a, f(a))$ has gradient $f'(a)$, a normal has gradient $-\frac{1}{f'(a)}$, and stationary points are where $f'(x) = 0$. Because the derivative is often printed in the stem as the target of a 'show that', always use the printed form in later parts even if your own proof failed; later marks are awarded for correct use of the given result. Write the rule you are using, differentiate each factor on its own line, then simplify to exactly the form requested ('in the form $\frac{p(x)}{q(x)}$' means a single fraction with the stated denominator).

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Formulas

$\frac{d}{dx}x^n = nx^{n-1}, n \in \mathbb{Q}$	in the booklet
$\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$	in the booklet
$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$	in the booklet
$y = uv \Rightarrow y' = uv' + vu'; y = \frac{u}{v} \Rightarrow y' = \frac{vu'-uv'}{v^2}$	in the booklet
$\frac{d}{dx}e^{kx} = ke^{kx}$, $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$, $\frac{d}{dx}\ln(ax+b) = \frac{a}{ax+b}$	learn it
$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$	learn it
$\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec x = \sec x\tan x$, $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x\cot x$, $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	in the booklet (AHL)
$\frac{d}{dx}a^x = a^x\ln a$, $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$	in the booklet (AHL)
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	in the booklet (AHL)
$\frac{d}{dx}\arctan(kx) = \frac{k}{1+k^2x^2}$, $\frac{d}{dx}\sec(kx) = k\sec kx\tan kx$	learn it

Worked examples

Example 1. Let $f(x) = \dfrac{e^{2x}}{2x+1}$, $x > -\dfrac{1}{2}$. (a) Show that $f'(x) = \dfrac{4xe^{2x}}{(2x+1)^2}$. [4] (b) Hence find the gradient of the graph of f at the point where $x = 1$. [2] (c) Write down the value of x at which the graph of f has a stationary point. [1]

1. (a) Use the quotient rule with $u = e^{2x}$, $v = 2x+1$. By the chain rule $u' = 2e^{2x}$, and $v' = 2$. (M1 for attempting the quotient rule, A1 for $u' = 2e^{2x}$.)
2. $f'(x) = \dfrac{(2x+1)\cdot 2e^{2x} - e^{2x}\cdot 2}{(2x+1)^2}$. (A1)
3. Factorise the numerator: $2e^{2x}(2x+1) - 2e^{2x} = 2e^{2x}(2x+1-1) = 4xe^{2x}$, so $f'(x) = \dfrac{4xe^{2x}}{(2x+1)^2}$. (A1, AG: the cancellation line must be shown.)
4. (b) Substitute $x = 1$ into the given derivative: $f'(1) = \dfrac{4\cdot 1\cdot e^2}{(2+1)^2} = \dfrac{4e^2}{9}$. (M1A1; leave exact on Paper 1.)
5. (c) $f'(x) = 0$ when the numerator is zero, $4xe^{2x} = 0$. Since $e^{2x} > 0$ for all x, this gives $x = 0$. (A1)

Answer: (a) $f'(x) = \dfrac{4xe^{2x}}{(2x+1)^2}$ (shown). (b) Gradient $= \dfrac{4e^2}{9}$. (c) $x = 0$.

Example 2. Let $f(x) = x^2e^{-x}$, $x \in \mathbb{R}$. (a) Find $f'(x)$, giving your answer in the form $p(x)e^{-x}$ where p is a polynomial. [3] (b) Show that $f''(x) = (x^2 - 4x + 2)e^{-x}$. [3] (c) Find the exact values of x for which $f''(x) = 0$. [2]

1. (a) Product rule with $u = x^2$, $v = e^{-x}$: $u' = 2x$ and, by the chain rule, $v' = -e^{-x}$. (M1 for the product rule, A1 for $v' = -e^{-x}$.)

2. $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$. (A1)
3. (b) Differentiate the simplified form again with $u = 2x - x^2$, $v = e^{-x}$: $u' = 2 - 2x$, $v' = -e^{-x}$. (M1)
4. $f''(x) = (2 - 2x)e^{-x} + (2x - x^2)(-e^{-x}) = (2 - 2x - 2x + x^2)e^{-x}$. (A1)
5. Collect terms: $f''(x) = (x^2 - 4x + 2)e^{-x}$. (A1, AG)
6. (c) Since $e^{-x} \neq 0$, solve $x^2 - 4x + 2 = 0$ by the quadratic formula: $x = \dfrac{4 \pm \sqrt{16 - 8}}{2} = \dfrac{4 \pm 2\sqrt{2}}{2}$. (M1)
7. So $x = 2 \pm \sqrt{2}$. (A1)

Answer: (a) $f'(x) = (2x - x^2)e^{-x}$. (b) $f''(x) = (x^2 - 4x + 2)e^{-x}$ (shown). (c) $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

Example 3. (AHL) (a) Let $f(x) = x \arctan(2x)$. Find $f'(x)$ and hence find the exact gradient of the graph of f at $x = \dfrac{1}{2}$. [5] (b) Let $g(x) = \ln(\sec x)$, $0 < x < \dfrac{\pi}{2}$. Show that $g'(x) = \tan x$. [3] (c) Write down $\dfrac{d}{dx}\left(3^x + \log_3 x\right)$. [2]

1. (a) Product rule with $u = x$, $v = \arctan(2x)$. $u' = 1$ and by the chain rule $v' = \dfrac{1}{1 + (2x)^2} \cdot 2 = \dfrac{2}{1 + 4x^2}$. (M1 for product rule, A1 for v' including the factor 2.)
2. $f'(x) = \arctan(2x) + \dfrac{2x}{1 + 4x^2}$. (A1)
3. At $x = \dfrac{1}{2}$: $f'\left(\frac{1}{2}\right) = \arctan 1 + \dfrac{1}{1 + 1} = \dfrac{\pi}{4} + \dfrac{1}{2}$. (M1A1; $\arctan 1 = \frac{\pi}{4}$ exactly, do not convert to a decimal.)
4. (b) Chain rule: outer function ln, inner function $\sec x$. $g'(x) = \dfrac{1}{\sec x} \cdot \dfrac{d}{dx}(\sec x)$. (M1)
5. From the booklet $\dfrac{d}{dx}\sec x = \sec x \tan x$, so $g'(x) = \dfrac{\sec x \tan x}{\sec x}$. (A1)
6. Cancel $\sec x$ (non-zero on the given domain): $g'(x) = \tan x$. (A1, AG)
7. (c) From the booklet, $\dfrac{d}{dx}3^x = 3^x \ln 3$ and $\dfrac{d}{dx}\log_3 x = \dfrac{1}{x \ln 3}$. (A1A1)

Answer: (a) $f'(x) = \arctan(2x) + \dfrac{2x}{1 + 4x^2}$; gradient at $x = \frac{1}{2}$ is $\dfrac{\pi}{4} + \dfrac{1}{2}$. (b) $g'(x) = \tan x$ (shown). (c) $3^x \ln 3 + \dfrac{1}{x \ln 3}$.

Common errors

- Dropping the inner derivative in a chain rule, writing $\frac{d}{dx}\sin 3x = \cos 3x$ or $\frac{d}{dx}\arctan(2x) = \frac{1}{1+4x^2}$. Always ask 'what is the derivative of the inside?' and write that factor explicitly before simplifying.

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- Reversing the order in the quotient rule or using a plus sign. Write u, u', v, v' in a table and copy the booklet formula $\frac{vu'-uv'}{v^2}$ exactly, denominator first.
- Sign errors with negative and fractional powers, for example $\frac{d}{dx}x^{-2} = -2x^{-3}$ not $-2x^{-1}$, and $\frac{d}{dx}\cos x = -\sin x$. Rewrite roots and reciprocals as powers before differentiating and subtract one from the index carefully.
- Jumping straight to the printed answer in a 'show that'. The final A1 (and the AG) depends on the factorising and cancelling steps being visible; write at least one intermediate line between the raw rule and the target form.
- Giving a decimal gradient on Paper 1 where an exact value such as $\frac{4e^2}{9}$ or $\frac{\pi}{4} + \frac{1}{2}$ is expected, or working in degrees so that $\frac{d}{dx}\sin x = \cos x$ no longer applies. Keep radians and exact forms unless an accuracy instruction is given.
- Differentiating an unsimplified $f'(x)$ to get $f''(x)$, producing a long expression full of errors. Simplify and factorise $f'(x)$ first, then differentiate the compact form.

Before the exam, can you...

- Before the exam, can you differentiate a polynomial with negative and fractional powers, and e^{kx}, $\sin(ax+b)$, $\cos(ax+b)$, $\ln(ax+b)$ without looking anything up?
- Can you state which rule a given expression needs (chain, product, quotient or a combination), set out u, u', v, v' and reach a factorised single-fraction form?
- Can you find $f''(x)$ from a simplified $f'(x)$, and explain why $f''(a) = 0$ alone does not prove a point of inflexion?
- (AHL) Can you write the derivatives of $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$, a^x, $\log_a x$, $\arcsin x$, $\arccos x$, $\arctan x$ with the correct signs, and apply the chain rule to composites such as $\arctan(3x)$ and $\sec(2x)$?
- Can you substitute $x = a$ into a derivative to obtain an exact gradient, and continue a question using the derivative printed in the stem even if your own 'show that' did not work?

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