

Conditional probability and Bayes' theorem: practice worksheet

IB Mathematics: analysis and approaches HL · SL 4.5-4.6, AHL 4.13 · 53 minutes · 44 marks

Question 1 (no calculator)

Events A and B are such that $P(A) = \frac{3}{5}$, $P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{17}{20}$.

- (a) Find $P(A \cap B)$. [2]
- (b) Find $P(A | B)$. [2]

Question 2 (no calculator)

A bag contains 3 red counters and 5 blue counters. Two counters are taken from the bag at random, one after the other, without replacement.

- (a) Find the probability that the two counters are the same colour. [2]
- (b) Given that the second counter is blue, find the probability that the first counter was red. [2]

Question 3 (no calculator)

For two events A and B , it is known that $P(A) = 0.7$, $P(B | A) = 0.2$ and $P(B | A') = 0.6$.

- (a) Find $P(B)$. [2]
- (b) Find $P(A | B')$. [3]
- (c) Determine whether events A and B are independent. [2]

Question 4 (calculator)

A bicycle manufacturer buys gear cables from three suppliers. Supplier P provides 50% of the cables, supplier Q provides 30% and supplier R provides the remaining 20%. From experience, 2% of cables from P , 4% of cables from Q and 5% of cables from R are faulty. A cable is selected at random from the manufacturer's stock. Let D be the event that the cable is faulty.

- (a) Find $P(D)$. [2]
- (b) Given that the cable is faulty, find the probability that it was supplied by Q . [3]
- (c) Given that the cable is not faulty, find the probability that it was supplied by R . [2]

Question 5 (calculator)

A blood test is used to screen for a condition that affects 3% of a population. If a person has the condition, the test gives a positive result with probability 0.96. If a person does not have the condition, the test gives a negative result with probability 0.92. A person is chosen at random from the population and tested. Let C be the event that the person has the condition and T be the event that the test result is positive.

- (a) Find $P(T)$. [2]
- (b) Given that the test result is positive, find the probability that the person has the condition. [2]
- (c) Given that the test result is negative, find the probability that the person has the condition. [2]
- (d) The health authority wants to improve the test so that, when the result is positive, the probability that the person has the condition is at least 0.5. The probability 0.96 of a positive result for a person with the condition is to remain unchanged, but the probability p of a negative result for a person without the condition is to be increased. Find the minimum value of p required. [3]

Question 6 (calculator)

A bakery uses two ovens. Oven A bakes 60% of the bakery's loaves and oven B bakes the remaining 40%. The probability that a loaf from oven A is underbaked is 0.05, and the probability that a loaf from oven B is underbaked is 0.12. Whether or not a loaf is underbaked is independent of all other loaves. Loaves are packed into boxes of six, all the loaves in any one box coming from the same oven, and 60% of the boxes come from oven A .

- (a) A loaf is chosen at random from the bakery's output. Find the probability that it is underbaked. [2]
- (b) Given that the loaf is underbaked, find the probability that it was baked in oven B . [2]
- (c)(i) A box of six loaves from oven B is chosen. Let X be the number of underbaked loaves in the box. Find $P(X = 2)$. [2]
- (c)(ii) Given that the box contains at most two underbaked loaves, find the probability that it contains at least one underbaked loaf. [3]
- (d) A box of six loaves is chosen at random from the bakery's output. Given that none of the loaves in the box is underbaked, find the probability that the box came from oven A . [4]

Markscheme

Question 1

(a)

use of addition rule: $P(A \cap B) = \frac{3}{5} + \frac{1}{2} - \frac{17}{20}$

M1

$$P(A \cap B) = \frac{5}{20} = \frac{1}{4}$$

A1

Accept 0.25.

(b)

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}}$$

M1

$$= \frac{1}{2}$$

A1

FT from an incorrect $P(A \cap B)$ in part (a) provided the value is between 0 and $\frac{1}{2}$.

Question 2

(a)

$$\frac{3}{8} \times \frac{2}{7} + \frac{5}{8} \times \frac{4}{7}$$

M1

$$= \frac{26}{56} = \frac{13}{28}$$

A1

A correctly labelled tree diagram with the two relevant branches identified earns the M1.

(b)

recognition of conditional probability in context: $P(\text{first red} | \text{second blue}) =$ **M1**

$$\frac{\frac{3}{8} \times \frac{5}{7}}{\frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{4}{7}} \left(= \frac{\frac{15}{56}}{\frac{35}{56}} \right)$$

$$= \frac{15}{35} = \frac{3}{7}$$

A1

Award M0 for a bare formula $P(A | B) = \frac{P(A \cap B)}{P(B)}$ with no values substituted.

Question 3

(a)

$$P(B) = 0.7 \times 0.2 + 0.3 \times 0.6$$

M1

$$= 0.14 + 0.18 = 0.32$$

A1

Accept $\frac{8}{25}$.

(b)

recognition that $P(A | B') = \frac{P(A \cap B')}{P(B')}$ with an attempt to find $P(A \cap B')$ via $0.7 \times$ **M1**

$$(1 - 0.2)$$

$$P(A \cap B') = 0.56 \text{ and } P(B') = 0.68$$

A1

$$P(A | B') = \frac{0.56}{0.68} = \frac{14}{17}$$

A1

FT from an incorrect $P(B)$ in part (a). Do not accept 0.7×0.68 as the numerator (assumes independence).

(c)

correct comparison, e.g. $P(B | A) = 0.2$ and $P(B) = 0.32$, or $P(A \cap B) = 0.14$ and **A1**

$$P(A)P(B) = 0.224$$

since these are not equal, A and B are not independent

R1

The R1 is dependent on a correct numerical comparison. Accept the equivalent comparison $P(B | A) \neq P(B | A')$ with a valid reason.

Question 4

(a)

$$P(D) = 0.5 \times 0.02 + 0.3 \times 0.04 + 0.2 \times 0.05$$

$$= 0.01 + 0.012 + 0.01 = 0.032$$

M1
A1

Accept 3.2% or $\frac{4}{125}$. A tree diagram with the three faulty branches identified earns the M1.

(b)

recognition of conditional probability in context: $P(Q | D) = \frac{P(Q \cap D)}{P(D)}$ **M1**

$$= \frac{0.3 \times 0.04}{0.032} \left(= \frac{0.012}{0.032} \right)$$

$$= 0.375$$

A1
A1

FT from an incorrect $P(D)$ in part (a). Accept $\frac{3}{8}$ or 37.5%.

(c)

$$P(R | D') = \frac{0.2 \times 0.95}{1 - 0.032} \left(= \frac{0.19}{0.968} \right)$$

$$= 0.196 \text{ (0.19628 ...)}$$

M1
A1

Award M0 if the numerator is 0.2×0.05 or the denominator is 0.032 (conditioning on the wrong event). FT from part (a).

Question 5

(a)

$$P(T) = 0.03 \times 0.96 + 0.97 \times 0.08$$

$$= 0.0288 + 0.0776 = 0.1064$$

M1
A1

Accept 0.106 or 10.6%.

(b)

$$P(C | T) = \frac{0.03 \times 0.96}{0.1064} \left(= \frac{0.0288}{0.1064} \right)$$

$$= 0.271 \text{ (0.27068 ...)}$$

M1
A1

FT from part (a). Accept 27.1%.

(c)

$$P(C | T') = \frac{0.03 \times 0.04}{1 - 0.1064} \left(= \frac{0.0012}{0.8936} \right)$$

$$= 0.00134 \text{ (0.001342 ...)}$$

M1
A1

FT from part (a). Award M0 for a denominator of 0.1064.

(d)

setting up $\frac{0.0288}{0.0288 + 0.97(1 - p)} = 0.5 \text{ (or } \geq 0.5)$ **M1**

$$0.97(1 - p) = 0.0288 \Rightarrow 1 - p = 0.02969 \dots$$

$$p = 0.970 \text{ (0.97031 ...)}$$

A1
A1

METHOD 2: solving the equation directly on the GDC earns M1 for the correct equation entered and A2 for the answer. Accept 97.0%; do not accept 0.97 unless the more accurate value is seen.

Question 6

(a)

$$0.6 \times 0.05 + 0.4 \times 0.12 = 0.03 + 0.048 = 0.078$$

M1
A1

Accept 7.8% or $\frac{39}{500}$.

(b)

$$P(B \mid \text{underbaked}) = \frac{0.4 \times 0.12}{0.078} \left(= \frac{0.048}{0.078} \right)$$

$$= \frac{8}{13} (0.615)$$

M1
A1

FT from part (a).

(c)(i)

recognition that $X \sim B(6, 0.12)$, e.g. $\binom{6}{2}(0.12)^2(0.88)^4$ or GDC binomial pdf
 $P(X = 2) = 0.130$ (0.12953 ...)

M1
A1

(c)(ii)

recognition of conditional probability in context: $P(X \geq 1 \mid X \leq 2) = \frac{P(1 \leq X \leq 2)}{P(X \leq 2)}$ **M1**
 $P(1 \leq X \leq 2) = 0.510$ (0.50950 ...) or $P(X \leq 2) = 0.974$ (0.97390 ...)
 $= 0.523$ (0.52315 ...)

A1
A1

Award M0 A0 A0 for $\frac{P(X \geq 1)}{P(X \leq 2)}$ (numerator not intersected with the conditioning event). Also accept the numerator written as $P(X \leq 2) - P(X = 0)$.

(d)

attempt to find the probability of no underbaked loaves for each oven: $P(\text{none} \mid A) = 0.95^6$ and $P(\text{none} \mid B) = 0.88^6$ **M1**
 $0.95^6 = 0.735$ (0.73509 ...) and $0.88^6 = 0.464$ (0.46440 ...) **A1**
 use of Bayes' theorem: $P(A \mid \text{none}) = \frac{0.6 \times 0.735 \dots}{0.6 \times 0.735 \dots + 0.4 \times 0.464 \dots} \left(= \frac{0.44106 \dots}{0.62682 \dots} \right)$ **M1**
 $= 0.704$ (0.70364 ...) **A1**

Award the first M1 for either binomial probability seen, including via GDC. FT the final A1 from incorrect values of 0.95^6 or 0.88^6 provided the Bayes structure is correct. Do not accept $0.6 \times (1 - 0.078)^6$ for the denominator (this ignores that all loaves in a box come from the same oven).