IB Maths: Analysis and Approaches HL *one-page formula sheet*

HL higher level only pietromeloni.com

Prior learning

Parallelogram	$A = bh$ (b base, h height)
Triangle	$A = \frac{1}{2}(bh)$ (b base, h height)
Trapezoid	$A = \frac{1}{2}(a+b)h$ (a, b parallel sides, h height)
Circle	$A = \pi r^2$ (r radius)
Circumference	$C = 2\pi r$ (r radius)
Cuboid	$V = lwh$ (l length, w width, h height)
Cylinder	$V = \pi r^2 h$ (r radius, h height)
Prism	$V = Ah$ (A area of cross-section, h height)
Cylinder, curved surface	$A = 2\pi rh$ (r radius, h height)
Distance (2D)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Midpoint (2D)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Topic 1: Number and algebra

1.2 Arithmetic: nth term	$u_n = u_1 + (n-1)d$
Arithmetic: sum	$S_n = \frac{n}{2}(2u_1 + (n-1)d)$; $S_n = \frac{n}{2}(u_1 + u_n)$
1.3 Geometric: nth term	$u_n = u_1 r^{n-1}$
Geometric: sum	$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$, $r \neq 1$

1.4 Compound interest

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

FV future value, PV present value, n number of years, k compounding periods per year, $r\%$ nominal annual rate of interest

1.5 Exponents, logs	$a^x = b \iff x = \log_a b$, $a > 0$, $b > 0$, $a \neq 1$

1.7 Exponents, logs

$$\log_a xy = \log_a x + \log_a y \qquad \log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^m = m \log_a x \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

1.8 Geometric: infinite sum	$S_\infty = \frac{u_1}{1 - r}$, $\lvert r \rvert < 1$

1.9 Binomial theorem

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \cdots + {}^nC_r a^{n-r} b^r + \cdots + b^n$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

HL 1.10 Combinations	${}^nC_r = \frac{n!}{r!(n-r)!}$
HL Permutations	${}^nP_r = \frac{n!}{(n-r)!}$
HL 1.12 Complex numbers	$z = a + bi$
HL 1.13 Polar, Euler form	$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\operatorname{cis}\theta$

HL 1.14 De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta$$

Topic 2: Functions

2.1 Straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
Gradient	$m = \frac{y_2 - y_1}{x_2 - x_1}$

2.6 Axis of symmetry

$$f(x) = ax^2 + bx + c \Rightarrow \text{axis of symmetry } x = -\frac{b}{2a}$$

2.7 Quadratic formula

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$$

Discriminant	$\Delta = b^2 - 4ac$

2.9 Exp and log functions

$$a^x = e^{x \ln a}; \quad \log_a a^x = x = a^{\log_a x}, \quad a, x > 0, \ a \neq 1$$

HL 2.12 Roots of $\sum_{r=0}^{n} a_r x^r = 0$	$\text{Sum} = \frac{-a_{n-1}}{a_n}$; $\text{product} = \frac{(-1)^n a_0}{a_n}$

Topic 3: Geometry and trigonometry

3.1 Distance (3D)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Midpoint (3D)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Right pyramid	$V = \frac{1}{3}Ah$ (A area of the base, h height)
Right cone	$V = \frac{1}{3}\pi r^2 h$ (r radius, h height)
Cone, curved surface	$A = \pi rl$ (r radius, l slant height)
Sphere volume	$V = \frac{4}{3}\pi r^3$ (r radius)
Sphere surface	$A = 4\pi r^2$ (r radius)
3.2 Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Triangle	$A = \frac{1}{2}ab\sin C$
3.4 Arc length	$l = r\theta$ (r radius, θ angle in radians)
Sector area	$A = \frac{1}{2}r^2\theta$ (r radius, θ angle in radians)
3.5 Tangent	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
3.6 Pythagorean	$\cos^2\theta + \sin^2\theta = 1$
Double angle	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
HL 3.9 Reciprocal	$\sec\theta = \frac{1}{\cos\theta}$ $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$
HL Pythagorean	$1 + \tan^2\theta = \sec^2\theta$ $\quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$
HL 3.10 Compound angle	$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
HL Double angle, tan	$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$
HL 3.12 Magnitude	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$ where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

HL 3.13 Scalar product

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta \text{ (θ angle between $\boldsymbol{v}$ and $\boldsymbol{w}$)}$$

HL Angle between vectors	$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
HL 3.14 Line: vector	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
HL Line: parametric	$x = x_0 + \lambda l$, $\quad y = y_0 + \lambda m$, $\quad z = z_0 + \lambda n$
HL Line: Cartesian	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
HL 3.16 Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$ where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$ (θ angle between $\boldsymbol{v}$ and $\boldsymbol{w}$)
HL Parallelogram	$A = \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ ($\boldsymbol{v}, \boldsymbol{w}$ two adjacent sides)
HL 3.17 Plane: vector	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
HL Plane: normal	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
HL Plane: Cartesian	$ax + by + cz = d$

Topic 4: Statistics and probability

4.2 Interquartile range	$\mathrm{IQR} = Q_3 - Q_1$
4.3 Mean $\bar{x}$	$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$
4.5 Probability	$\mathrm{P}(A) = \frac{n(A)}{n(U)}$
Complementary	$\mathrm{P}(A) + \mathrm{P}(A') = 1$
4.6 Combined	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
Mutually exclusive	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$
Conditional	$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
Independent	$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
4.7 Expected value (discrete)	$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$
4.8 Binomial $X \sim \mathrm{B}(n, p)$	$\mathrm{E}(X) = np$ $\quad \mathrm{Var}(X) = np(1-p)$
4.12 Standardized normal	$z = \frac{x - \mu}{\sigma}$

HL 4.13 Bayes' theorem

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$$

$$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$$

HL 4.14 Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
HL Standard deviation	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}}$

HL Linear transformation

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

HL Expected value (continuous)	$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$
HL Variance	$\mathrm{Var}(X) = \mathrm{E}(X - \mu)^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

HL Variance (discrete)

$$\mathrm{Var}(X) = \sum (x - \mu)^2 \mathrm{P}(X = x) = \sum x^2 \mathrm{P}(X = x) - \mu^2$$

HL Variance (continuous)

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$$

Topic 5: Calculus

5.3 Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
5.5 Integral of x^n	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$, $\quad n \neq -1$
Area, $f(x) > 0$	$A = \int_a^b y\,\mathrm{d}x$
5.6 Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
Derivative of e^x	$f(x) = e^x \Rightarrow f'(x) = e^x$
Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
Product rule	$y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
5.9 Acceleration	$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$
Distance travelled	$\text{distance} = \int_{t_1}^{t_2} \lvert v(t) \rvert\,\mathrm{d}t$
Displacement	$\text{displacement} = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t$
5.10 Standard integrals	$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x \rvert + C$ $\quad \int \sin x\,\mathrm{d}x = -\cos x + C$ $\int \cos x\,\mathrm{d}x = \sin x + C$ $\quad \int e^x\,\mathrm{d}x = e^x + C$
5.11 Area to x-axis	$A = \int_a^b \lvert y \rvert\,\mathrm{d}x$
HL 5.12 First principles	$y = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$
HL 5.15 Derivatives	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ $f(x) = \sec x \Rightarrow f'(x) = \sec x\tan x$
HL Derivatives	$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x\cot x$ $f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$ $f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$
HL Derivatives	$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x\ln a}$ $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1 - x^2}}$
HL Derivatives	$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1 - x^2}}$ $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1 + x^2}$
HL Standard integrals	$\int a^x\,\mathrm{d}x = \frac{1}{\ln a}a^x + C$ $\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C$, $\quad \lvert x \rvert < a$

HL 5.16 Integration by parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \quad \text{or} \quad \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

HL 5.17 Area to y-axis	$A = \int_a^b \lvert x \rvert\,\mathrm{d}y$
HL Volume of revolution	$V = \int_a^b \pi y^2\,\mathrm{d}x$ or $V = \int_a^b \pi x^2\,\mathrm{d}y$

HL 5.18 Euler's method

$$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h \text{ (h constant step length)}$$

HL Integrating factor, $y' + P(x)y = Q(x)$	$e^{\int P(x)\,\mathrm{d}x}$
HL 5.19 Maclaurin series	$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

HL Maclaurin: special

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \qquad \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$